Recursive Functions

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Introduction

As we have seen, many functions can naturally be defined in terms of other functions.

```
fac  :: Int → Int
fac n = product [1..n]
```

fac maps any integer n to the product of the integers between 1 and n.

Expressions are <u>evaluated</u> by a stepwise process of applying functions to their arguments.

```
fac   :: Int → Int
fac n = product [1..n]
```

For example:

```
  fac 4

= product [1..4]

= product [1,2,3,4]

= 1*2*3*4

= 24
```

Recursive Functions

In Haskell, functions can also be defined in terms of themselves. Such functions are said to be *recursive*.

```
fac :: Int -> Int
fac 1 = 1
fac n = n * fac (n-1)
```

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```
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```

Example

```
Fac 1 = 1
fac n = n * fac (n-1)
```

```
   fac 3
=  3 * fac 2
=  3 * (2 * fac 1)
=  3 * (2 * (1))
=  3 * 2
=  6
```

- The recursive definition diverges on integers < 0 because the base case is never reached:

```
> fac (-1)

Error: “Recurse forever”
```

Why is Recursion Useful?

- Some functions, such as factorial, are simpler to define in terms of other functions.

- As we shall see, however, many functions can naturally be defined in terms of themselves.

- Properties of functions defined using recursion can be proved using the simple but powerful mathematical technique of induction.

Recursion on Lists

Recursion is not restricted to numbers, but can also be used to define functions on lists.

```
product        :: Num a ⇒ [a] → a
```

```
product [ ]     =  1
product (x : xs) =  x * product xs
```

Recursion on Lists

Recursion is not restricted to numbers, but can also be used to define functions on lists.

```
product        :: Num a ⇒ [a] → a
product []      = 1
product (n:ns) = n * product ns
```

product maps the empty list to 1, and any non-empty list to its head multiplied by the product of its tail.

Example

```
product        :: Num a ⇒ [a] → a
product []       = 1
product (n:ns) = n * product ns
```

```
    product [2,3,4]

=   2 * product [3,4]

=   2 * (3 * product [4])

=   2 * (3 * (4 * product []))

=   2 * (3 * (4 * 1))

=   24
```

Using the same pattern of recursion as in product we can define the length function on lists.

```
length :: [a] -> Int
length []     = 0
length (_:xs) = 1 + length xs
```

Using the same pattern of recursion as in product we can define the length function on lists.

```
length        :: [a] → Int
length []      = 0
length (_:xs) = 1 + length xs
```

length maps the empty list to 0, and any non-empty list to the successor of the length of its tail.

```
length        :: [a] → Int
length []      = 0
length (_:xs) = 1 + length xs
```

Example

```
    length [1,2,3]

=   1 + length [2,3]

=   1 + (1 + length [3])

=   1 + (1 + (1 + length []))

=   1 + (1 + (1 + 0))

=   3
```

```
(:) :: a -> [a] -> [a]
(++)
```

Using a similar pattern of recursion we can define the reverse function on lists.

```
reverse :: [a] -> [a]
reverse []     = []
reverse (x:xs) = (reverse xs) ++ [x]
```

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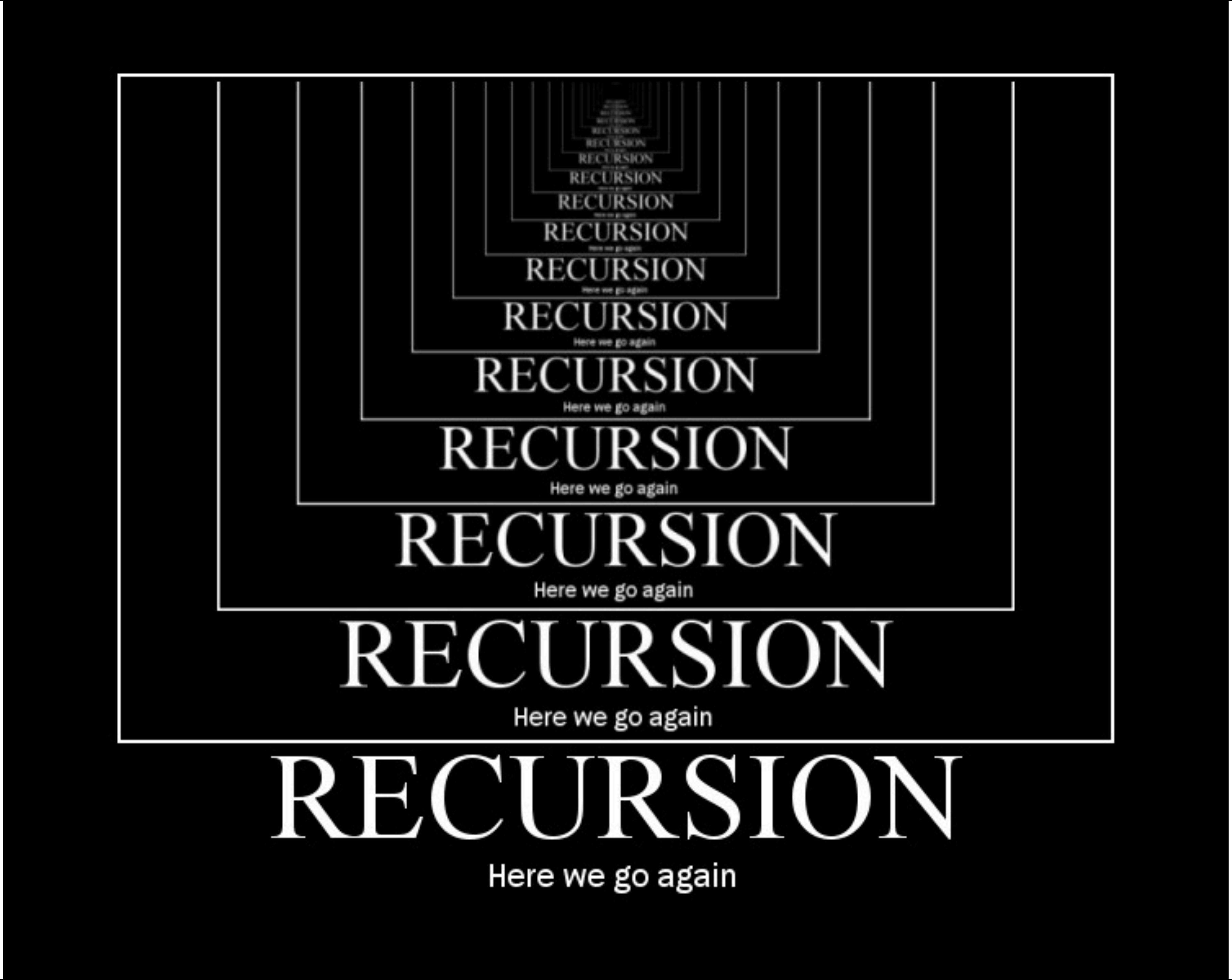
```
reverse        :: [a] → [a]
reverse []      = []
reverse (x:xs) = reverse xs ++ [x]
```

reverse maps the empty list to the empty list, and any non-empty list to the reverse of its tail appended to its head.

```
reverse        :: [a] → [a]
reverse []      = []
reverse (x:xs) = reverse xs ++ [x]
```

Example

```
    reverse [1,2,3]

=   reverse [2,3] ++ [1]

=   (reverse [3] ++ [2]) ++ [1]

=   ((reverse [] ++ [3]) ++ [2]) ++ [1]

=   (([] ++ [3]) ++ [2]) ++ [1]

=   [3,2,1]
```


RECURSION
Here we go again

Multiple Arguments

Functions with more than one argument can also be defined using recursion.

- For example, zipping the elements of two lists:

```
zip :: [a] -> [b] -> [(a, b)]
zip [] _ = []
zip _ [] = []
zip (x:xs) (y:ys) = [(x, y)] ++ zip xs ys
```

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- For example, zipping the elements of two lists:

```
zip             :: [a] → [b] → [(a,b)]
zip []     _     = []
zip _      []    = []
zip (x:xs) (y:ys) = (x,y) : zip xs ys
```

- Remove the first n elements from a list:

```
drop        :: Int → [a] → [a]
```

```
drop 0 xs = xs
drop n [] = []
drop n (x:xs) = drop (n-1) xs
```

- Appending two lists:

```
(++)        :: [a] → [a] → [a]
```

```
(++) xs [] = xs
(++) [] y = y
```

```
(++) (x:xs) ( ys) =
    x: ( xs++ys )
```

- Remove the first n elements from a list:

```
drop          :: Int → [a] → [a]
drop 0 xs      = xs
drop _ []      = []
drop n (_:xs) = drop (n-1) xs
```

- Appending two lists:

```
(++)          :: [a] → [a] → [a]
[]     ++ ys = ys
(x:xs) ++ ys = x : (xs ++ ys)
```

Quick Sort

Quick Sort

The quicksort algorithm for sorting a list of values can be specified by the following two rules:

- The empty list is already sorted;
- Non-empty lists can be sorted by sorting the tail values $\leq$ the head, sorting the tail values $>$ the head, and then appending the resulting lists on either side of the head value.

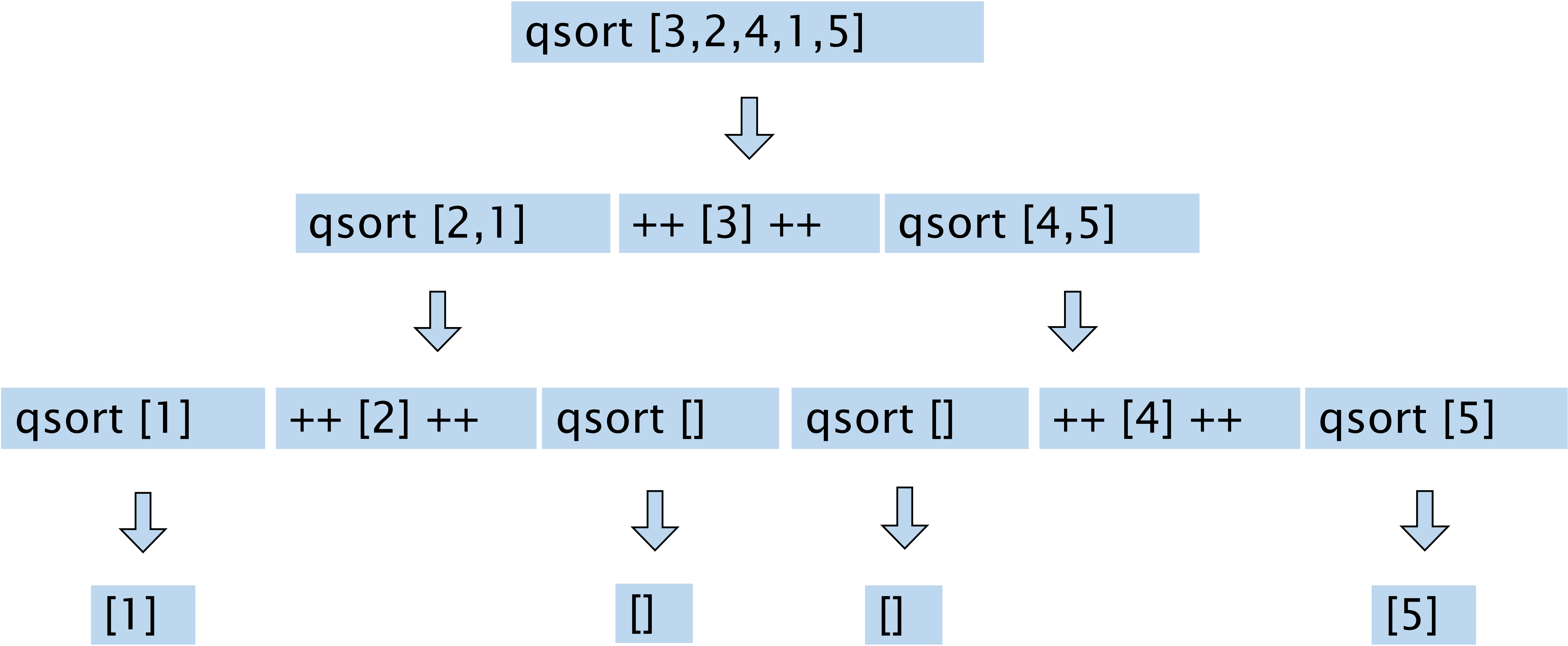
Using recursion, this specification can be translated directly into an implementation:

Using recursion, this specification can be translated directly into an implementation:

```
qsort        :: Ord a ⇒ [a] → [a]
qsort []      = []
qsort (x:xs) =
   qsort smaller ++ [x] ++ qsort larger
   where
      smaller = [a | a ← xs, a ≤ x]
      larger  = [b | b ← xs, b > x]
```

- This is probably the simplest implementation of quicksort in any programming language!

For example (abbreviating qsort as q):



qsort [3,2,4,1,5]

qsort [2,1] ++ [3] ++ qsort [4,5]

qsort [1] ++ [2] ++ qsort [] qsort [] ++ [4] ++ qsort [5]

[1] [] [] [5]

Exercises

(1) Without looking at the standard prelude, define the following library functions using recursion:

z Decide if all logical values in a list are true:

```
and :: [Bool] → Bool
```

z Concatenate a list of lists:

```
concat :: [[a]] → [a]
```

- Produce a list with n identical elements:

```
replicate :: Int → a → [a]
```

- Select the nth element of a list:

```
(!!) :: [a] → Int → a
```

- Decide if a value is an element of a list:

```
elem :: Eq a ⇒ a → [a] → Bool
```

(2) Define a recursive function

```
merge :: Ord a ⇒ [a] → [a] → [a]
```

that merges two sorted lists of values to give a single sorted list. For example:

```
> merge [2,5,6] [1,3,4]

[1,2,3,4,5,6]
```

(3) Define a recursive function

`msort :: Ord a ⇒ [a] → [a]`

that implements merge sort, which can be specified by the following two rules:

- Lists of length ≤ 1 are already sorted;
- Other lists can be sorted by sorting the two halves and merging the resulting lists.