Mathematatic Induction

Objective

- Induction on Integers
- Induction on Structures

Prove

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}, \qquad \forall n \in \mathbb{N}$$

for all

`$\mathbb{N}$`

$\mathbb{N}$: the set of natural integers, $\{1, 2, 3, \ldots\}$

$\forall n \in \mathbb{N}$: for every integer n in $\mathbb{N}$.

`\in`

"`\in`"

$$\sum_{i=1}^{i=n} i = 1 + 2 + \cdots + n$$

First attempt

Prove

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}, \qquad \forall n \in \mathbb{N}$$

$n = 1$ LHS = 1 RHS = $\frac{1 \times (1+1)}{2}$ = 1, LHS=RHS

$n = 2$ LHS = 1+2 = 3 RHS = $\frac{2 \times 3}{2}$ = 3 = LHS

LHS = RHS

$n = 3$

$n = 4$

$\cdots\cdots$

You will never finish the proof … !

Mathematic Induction

- **Base step**
  - Prove the identity for a particular n value (such as $n = 1$, depending on your goal,).
- **Inductive hypothesis**
  - Assuming the identity holds for all $n \leq k$,
- **Induction step**
  - Prove the identity also holds for $n = k + 1$.

$$\frac{ak}{2} + \frac{kb}{2} = \frac{ak+kb}{2} = \frac{k(a+b)}{2}$$

Prove

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}, \qquad \forall n \in \mathbb{N}$$

Base step: if $n=1$ LHS $=1$. RHS $= \frac{1 \times (1+1)}{2} = 1$

LHS = RHS

Inductive hypothesis:

For a given K, $1+2+\cdots+n = \frac{n(n+1)}{2}$ holds for all $n \le K$.

Induction step:

$\text{RHS} = \frac{(K+1)(K+2)}{2}$

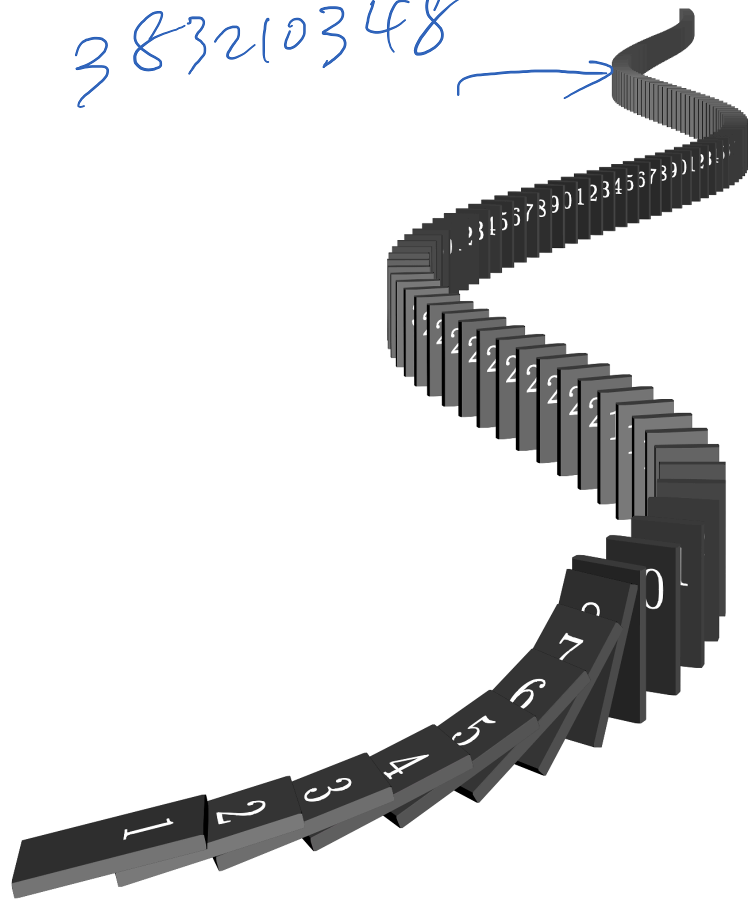
Want to prove the equation holds for $n=K+1$.

$$\text{LHS} = \underline{1+2+\cdots+K} + (K+1) = \frac{K(K+1)}{2} + \frac{(K+1)2}{2} = \frac{(K+1)(K+2)}{2}$$

Are we done?

- Yes! But why? (I haven't proved the theorem for many particular n such as 383210348 yet. Am I really done?)

383210348



Infinity?

- Does this proof show that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ even when $n = \infty$?

No.

if $k = \infty$

$\infty + 1 \triangleq \infty$

Prove $\forall n \in \mathbb{N}$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

$\forall n \in \mathbb{N}$ means "for every integer n in $\{1, 2, 3, \ldots\}$."

Base step:

$LHS = 1 = RHS = \frac{1 \times 2 \times 3}{6} = 1$

Prove $\forall n \in \mathbb{N}$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Inductive hypothesis:

There exists some k such that if $n \leq k$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Induction step:

want to show $1^2 + \cdots + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{2}$

Prove $\forall n \in \mathbb{N}$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Induction step:

$$LHS = 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

If $n = k + 1$,

$$= \frac{1}{6}\left[k(k+1)(2k+1) + 6\cdot(k+1)^2\right]$$

$$= \frac{(k+1)}{6}\left[k(2k+1) + 6(k+1)\right]$$

$$= \frac{(k+1)}{6}\left[2k^2 + k + 6k + 6\right] = \frac{(k+1)}{6}(2k^2 + 7k + 6)$$

$$(k+2)(2k+3)$$
$$= 2k^2 + 3k + 4k + 6$$
$$= 2k^2 + 7k + 6$$

$$
\begin{array}{r}
2k + 3 \\
k+2 \,\overline{)\, 2k^2 + 7k + 6} \\
-\;\; 2k^2 + 4k \\
\hline
3k + 6 \\
3k + 6 \\
\hline
0
\end{array}
$$

$2k(k+2)$

$3(k+2)$

0 ← Remainder.

(f a b) * c

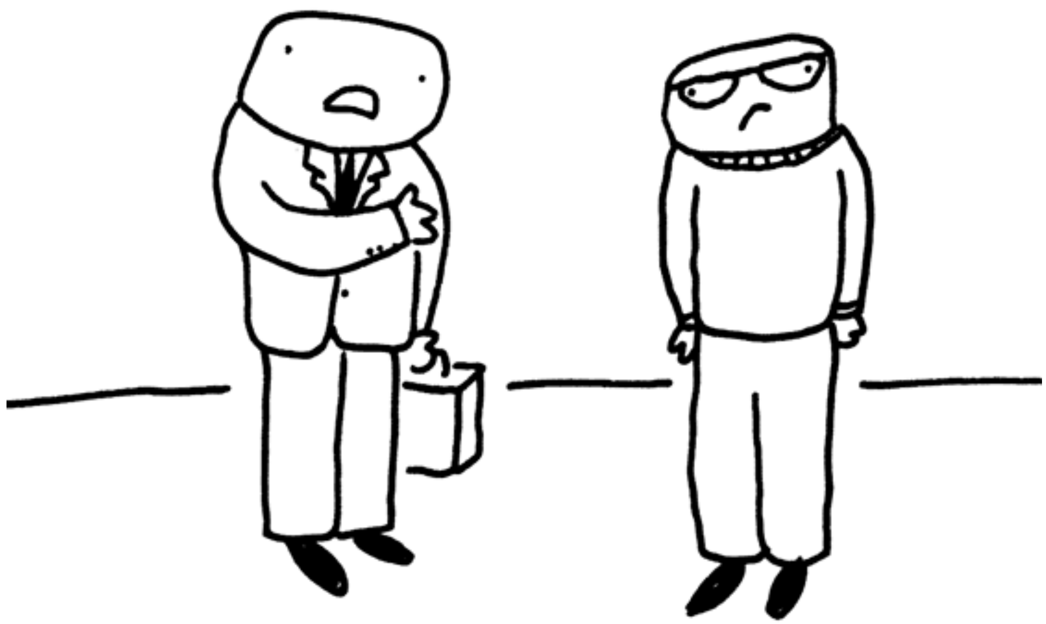
f a b c Not Right

f (a, b) x c

f a b * c

correct!

i don't care if you're
a mathematician or not...
the judge is going to
need more proof than "Q.E.D."



Toothpaste For Dinner.com

Prove $\forall n \in \mathbb{N}$,

$$n! \leq n^n.$$

Base step: if $n=1$

LHS $=1$

RHS $=1$

Prove $\forall n \in \mathbb{N}$,

$$n! \leq n^n.$$

Inductive hypothesis:

$\exists K$, for all $n \leq K$, $n! \leq n^n$

$RHS = (K+1)^{(K+1)}$

Induction step:

when $n = K+1$,

$LHS = (K+1)! = K!\,(K+1) \leq K^K (K+1) \leq (K+1)^K (K+1)$

$= (K+1)^{(K+1)} = RHS$

Induction over Structures

Structural Induction

1. Prove the statement for the base cases.

2. State the hypothesis

3. Prove the statement for every inductive rules

Structural Induction

The set S is defined as follows,

(1) $3 \in S$.

(2) If $x, y \in S$, then $x + y \in S$.

Prove $S \subseteq \{3n \mid n \in \mathbb{Z}^+\}$.

Base Step

The set S is defined as follows,

(1) $3 \in S$.

(2) If $x, y \in S$, then $x + y \in S$.

Prove $S \subseteq \{3n \mid n \in \mathbb{Z}^+\}$.

- We want to show $3 \in \{3n|n \in \mathbb{Z}^+\}$.

Proof:

Let $n = 1 \in \mathbb{Z}^+$, $3 = 3 * 1 = 3n \in \{3n|n \in \mathbb{Z}^+\}$.

Thus, $3 \in \{3n|n \in \mathbb{Z}^+\}$.

Inductive Hypothesis

The set S is defined as follows,

(1) $3 \in S$.

(2) If $x, y \in S$, then $x + y \in S$.

Prove $S \subseteq \{3n \mid n \in \mathbb{Z}^+\}$.

If $x, y \in S$, then $x, y \in \{3n | n \in \mathbb{Z}^+\}$.

Induction Step

The set S is defined as follows,

(1) $3 \in S$.

(2) If $x, y \in S$, then $x + y \in S$.

Prove $S \subseteq \{3n \mid n \in \mathbb{Z}^+\}$.

- We want to show:

$$\text{If } x, y \in S, \text{ then } x + y \in \{3n | n \in \mathbb{Z}^+\}.$$

Proof:

Since $x, y \in S$, by the inductive hypothesis, $x, y \in \{3n | n \in \mathbb{Z}^+\}$. Hence, there exist $n_1, n_2 \in \mathbb{Z}^+$ such that $x = 3n_1, y = 3n_2$. Therefore, $x + y = 3(n_1 + n_2) \in \{3n | n \in \mathbb{Z}^+\}$ because $n_1 + n_2 \in \mathbb{Z}^+$.

Exercise

- Prove $\forall\, n \in \mathbb{N}$,

$$3 + 3^2 + 3^3 + 3^4 + \ldots + 3^n = \frac{3^{n+1} - 3}{2}$$