Probability (2)

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Review

- A finite **probability space** is denoted by (S, P) where
  - S is a finite set (the *sample space*), and
  - P is a function $S \to [0,1]$ (the *probability measure*) such that

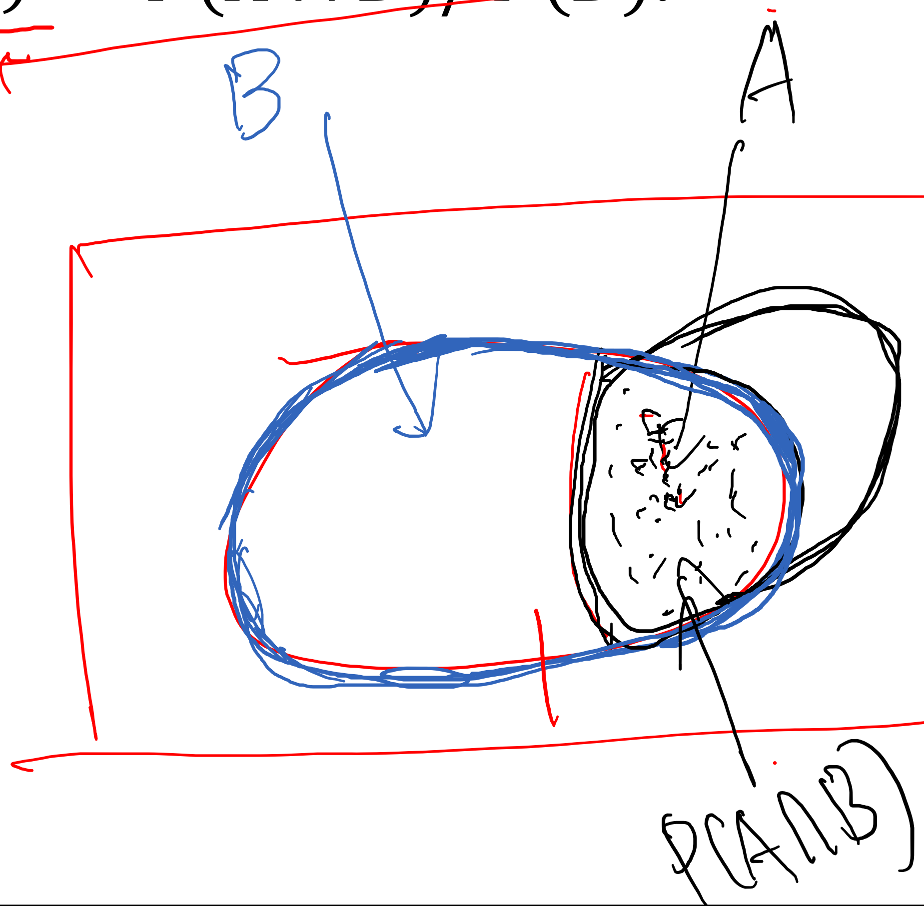
$$\sum_{x \in S} P(x) = 1$$

Whenever hearing "probability", make sure that you are clear what the probability space is: *what is the sample space and what is the probability measure on it*.

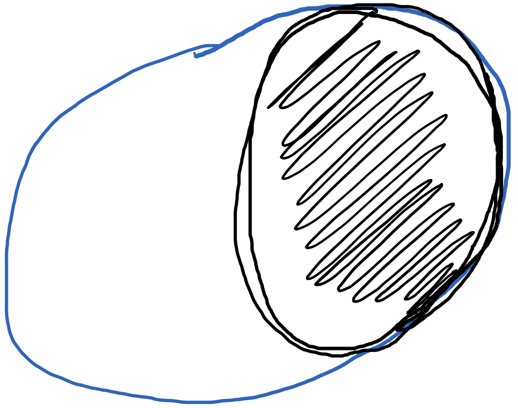
Conditional Probability

$P(A \cap B) = P(A|B)\,P(B)$

If $B \neq \phi$, the probability of even A conditioned on the fact that B happens, $P(A|B) = P(A \cap B)/P(B)$.



$P(A|B)$



Independent Events

Events A and B are **independent** if

$$P(A|B) = P(A).$$

An equivalently definition of independent events A and B:

$$P(A \cap B) = P(A)\,P(B).$$

$P(A \cap B) = 0$

$P(A) = \frac{1}{2}$ $P(B) = \frac{1}{2}$

$P(A \cap B)$

$\neq P(A) \cdot P(B)$

tossing a fair coin 1 time

A = { the sum of the two outcomes is 0 }

B = { " " " " " " " " " 1 }

52

- Drawing (*with replacement*) two cards from a standard deck. Let

E = {the first card is a King}

F = {the second card is a King}

$$P(E \wedge F) = \frac{4}{52} \cdot \frac{4}{52}$$

$$P(E) = P(F) = \frac{4}{52}$$

E and F are independent.

- Drawing (*without replacement*) two cards from a standard deck. Let

$$E = \{\text{the first card is a King}\}$$

$$F = \{\text{the second card is a King}\}$$

E and F are not independent

$$P(E) = \frac{4}{52}$$

$$P(F) = \begin{cases} \frac{3}{51}, & \text{if } E \text{ occurs} \\ \frac{4}{51}, & \text{if } E \text{ doesn't occur.} \end{cases}$$

$$P(F) = P(F|E)\cdot P(E) + P(F|\bar{E})P(\bar{E})$$

$$= \frac{3}{51}\cdot\frac{4}{52} + \frac{4}{51}\cdot\frac{48}{52} = \frac{12+192}{51\times 52} = \frac{204}{51\times 52}$$

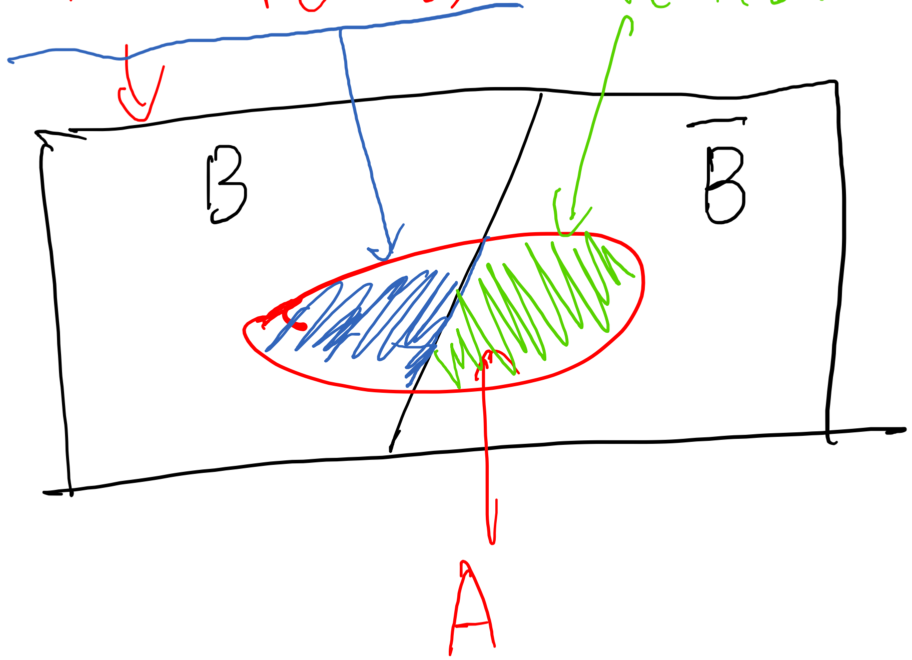
$$P(E \cap F) = P(F|E)\,P(E) = \frac{4}{52}\times\frac{3}{51} = \frac{1}{221} \neq P(E)P(F)$$

Bayes's Formula (1)

For any two events A and B,

$$P(A) = P(A|B) * P(B) + P(A|\overline{B}) * P(\overline{B})$$

$P(B) \cdot P(A|B)$ $\quad$ $P(A|\overline{B}) \cdot P(\overline{B})$



$$P(A|B) = \frac{P(A) - P(A|\overline{B}) \cdot P(\overline{B})}{P(B)}$$

$$P(B) = P(B|A) \cdot P(A) + P(B|\overline{A}) P(\overline{A})$$

Bayes's Formula (2)

For any two events A and B,

$$P(A|B) = \frac{P(B|A)*P(A)}{P(B|A)*P(A)+P(B|\overline{A})*P(\overline{A})}$$

$P(A) - P(A|\bar{B})P(\bar{B}) = P(B|A)\cdot P(A)$

$(1 - P(B|A))\, P(A) = P(A|\bar{B})P(\bar{B})$

$P(\bar{B}|A)\, P(A) = P(A|\bar{B})P(\bar{B})$

$P(A \cap \bar{B}) = P(A \cap \bar{B})$

$$P(A|B) = \frac{P(A) - P(A|\bar{B})P(\bar{B})}{P(B|A)P(A) + P(B|\bar{A})P(\bar{A})}$$

$$= \frac{P(B|A)\cdot P(A)}{P(B|A)\cdot P(A) + P(B|\bar{A})\cdot P(\bar{A})}$$

$\blacksquare$

Exercise 1

- Two urns:

    Urn #1 has 10 gold coins and 5 silver coins

    Urn #2 has 2 gold coins and 8 silver coins

  First randomly pick an urn then randomly pick a coin from the urn.

  What is the probability it is a gold coin?

$$P(G|B) = \frac{10}{10+5} = \frac{2}{3}$$

$$P(B) = \frac{1}{2} \quad P(\bar{B}) = \frac{1}{2}$$

$$P(G|\bar{B}) = \frac{2}{2+8} = \frac{1}{5}$$

$G = \{$the coin picked is golden$\}$

$B = \{$urn 1 was picked$\}$

$\bar{B} = \{$urn 2 was picked$\}$

$$P(G) = P(G|B)\cdot P(B) + P(G|\bar{B})P(\bar{B})$$

$$= \frac{2}{3}\cdot\frac{1}{2} + \frac{1}{5}\cdot\frac{1}{2} = \frac{1}{3} + \frac{1}{10} = \frac{13}{30}$$

$B = \{\text{urn \#1 was picked}\}$ $\quad P(G|B) = \frac{10}{15} = \frac{2}{3}$ $\quad P(G|\bar{B}) = \frac{1}{5}$

Exercise 2

- Two urns:

  Urn #1 has 10 gold coins and 5 silver coins

  Urn #2 has 2 gold coins and 8 silver coins

  First randomly pick an urn then randomly pick a coin from the urn. It turns out that the coin is golden. What is the probability that urn #1 was picked?

$$P(B|G) = \frac{P(G|B)P(B)}{P(G|B)P(B) + P(G|\bar{B})P(\bar{B})}$$

$$= \frac{2/3 \cdot \frac{1}{2}}{2/3 \cdot \frac{1}{2} + \frac{1}{5} \cdot \frac{1}{2}} = \frac{1/3}{13/30} = \frac{10}{13}$$

Random Variable

A random variable is a function $X: S \rightarrow \mathbb{R}$ (from the sample space to the reals)

coin-tossing.

$S = \{H, T\}$

$X(H) = 1$

$X(T) = 0$

Shooting game.

$X(v) = v, \ v \in [1, 10]$

Expectation of a Random Variable

- The expected value of a random variable X, denoted by $\mathbf{E}[X]$, is defined as

$$\mathbf{E}[X] = \sum_{s \in S} P(s)X(s)$$

- Example - Fair coin tossing: define X(H)=1, X(T)=0.

$$E(X) = P(H)\,X(H) + P(T)\cdot X(T)$$

$$= \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 0 = \frac{1}{2}$$

Shooting competition

$S = \{5, 6, 7, 8, 9, 10\}$

$X = id$

	10	9	8	7	6	5
James	45%	30%	27%	13%	1%	0
Venny	55%	18%	17%	5%	3%	2%

Who is more likely to win in a competition?

James: $E(X) = 45\% \cdot 10 + 30\% \cdot 9 + 27\% \cdot 8 + 13\% \cdot 7 + 1\% \cdot 6$

$= 4.5 + 2.7 + 2.16 + 0.91 + 0.06$

V

Venny: $E(X) = 55\% \cdot 10 + 18\% \cdot 9 + 17\% \cdot 8 + 5\% \cdot 7 + 3\% \cdot 6 + 2\% \cdot 5$

$= 5.5 + 1.62 + 1.36 + 0.35 + 0.18 + 0.1$

More examples

- What is the expected outcome of rolling a dice?

$$E(X) = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\times 3 + \frac{1}{6}\times 4 + \frac{1}{6}\times 5 + \frac{1}{6}\cdot 6$$

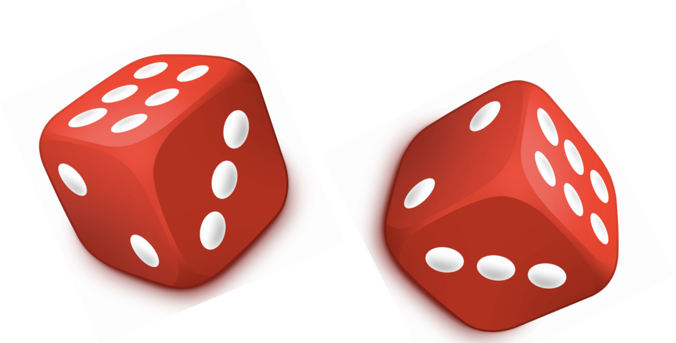
$$= \frac{1}{6}(1+6)\times 6/2 = \frac{1}{6}\cdot 21 = \frac{7}{2}$$

- Rolling a fair dice, what is the expectation of the *square* of the outcomes?

$$E(X^2) = \sum_{s \in S} P(s) \cdot X^2(s) = \frac{1}{6} \sum_{s \in S} X^2(s)$$

$$= \frac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) = 91/6$$

More examples



- How about rolling a dice ***twice***? what is the expection of the sum of the two outcomes?

$X(s)$	2	3	4	5	6	7	8	9	10	11	12
$P(s)$	1/36	1/18	1/12	1/9	5/36	1/6	5/36	1/9	1/12	1/18	1/36

$$E(X) = 2\times\frac{1}{36} + 3\times\frac{1}{18} + \cdots + 12\times\frac{1}{36} = 7.$$

Linearity of Expectation

- For random variables X and Y (which may be dependent),

$$E[X+Y] = E[X] + E[Y]$$

- More generally, for random variables $X_1, X_2, \dots, X_n$ and constants $c_1, c_2, \dots, c_n$,

$$E[c_1 X_1 + \cdots + c_n X_n] = c_1 E[X_1] + \cdots + c_n E[X_n]$$

Better way

- Expected outcome of rolling a dice ***twice***? what is the ~~expectation~~ of the sum of the two outcomes.

X_1 : the random variable of the first roll.

X_2 : - - - - - - - - second roll

$$E[X_1+X_2] = E[X_1] + E[X_2] = 7/2 + 7/2 = 7$$

Exchanging Gifts

- At a Christmas party, n friends each bought a gift box and mixed them together. Later, each person randomly draw a gift box from the pile. On average, how many people will get back their own gift?

if $n = 3$.

$3! = 6$

1 2 3	1 3 2	2 1 3	2 3 1	3 1 2	3 2 1
1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3

$$3 \times \frac{1}{6} + 1 \times \frac{1}{6} + 1 \times \frac{1}{6} + 0 \times \frac{1}{6} + 0 \times \frac{1}{6} + 1 \times \frac{1}{6}$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

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$$X_i = \begin{cases} 1, & \text{the } i^{\text{th}} \text{ person gets back his/her own gift} \\ 0, & \text{otherwise} \end{cases}$$

$$E(X_i) = 1 \times \frac{1}{n} + 0 \times \frac{n-1}{n} = \frac{1}{n}$$

Binary random variable like X_i is called an ***indicator*** random variable.

$$E[X_1 + X_2 + \cdots + X_n] = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \frac{1}{n} = 1$$