Probability (1)

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Definition

- A finite **probability space** is denoted by (S, P) where
  - S is a finite set (the *sample space*), and
  - P is a function $S \to [0,1]$ (the *probability measure*) such that

$$\sum_{x \in S} P(x) = 1$$

Whenever hearing "probability", make sure that you are clear what the probability space is: *what is the sample space and what is the probability measure on it*.

Probability Space (S, P)

- An *outcome* is a point in S.
- An *event* is a subset of S.

Throw a dice

$S = \{1, 2, 3, 4, 5, 6\}$

$E_1 = \{$all the outcomes less than $4\}$

$= \{1, 2, 3\}$

$E_2 = \{1\}$

Uniform Distribution

- Every point in S is *equiprobable*

$$P(a) = \frac{1}{|S|}$$

e.g. fair dice of 6 faces.

$$P(X=1) = \frac{1}{6} = P(X=2) = P(X=3) = \cdots$$

Probabilities of events

- Let A be an event of probability space (S, P). The probability of event A is

$$P(A) \coloneqq \sum_{a \in A} P(a).$$

Assume ~~fair~~ dice.

$$P(E_1) = \sum_{a \in \{1,2,3\}} P(a) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

Examples

- Toss a fair dice twice, what is the probability that the two outcomes add up to 5?

$S = \{(1,1), (1,2), (2,1), (2,2), \ldots (6,6)\}$

$|S| = 6 \times 6 = 36$

$E = \{(1,4), (2,3), (3,2), (4,1)\}$

$P(E) = \frac{4}{36} = \frac{1}{9}$

Examples

- Toss a fair dice three times, what is the probability that the sum of the outcomes is less than 10?

$$S = \{(i,j,k) \mid 1 \le i \le 6,\ 1 \le j \le 6,\ 1 \le k \le 6\}$$

$$|S| = 6^3$$

$$E = \{(i,j,k) \mid i+j+k < 10\} = \{\overbrace{(1,1,1),\ (1,1,2), \cdots (1,1,7)}^{7},\ \underbrace{(1,2,1) \cdots (1,2,6)}_{6},\ \underbrace{(1,3,1) \cdots (1,3,5)}_{5}$$

$(1,4,1), \cdots (1,4,4)$ 4

$(1,5,1), \cdots (1,5,3)$ 3

$(1,6,1), \cdots (1,6,2)$ 2

$(2,1,1), \cdots (2,1,6)$ 6

$(2,2,1), \cdots (2,2,5)$ 5

$(2,3,1), \cdots (2,3,4)$ 4

$(2,4$ ⋮

$(2,5,$

$(2,6$ 1

$(6,1,1) \cdots (6,1,2)$

$(6,2,1)$

$$|E| = (7+6+\cdots+2) + (6+5+\cdots+1) + (5+4+\cdots+1) + (4+3+\cdots+1) + (3+2+1) + (2+1)$$

$$= \frac{(2+7)\times 6}{2} + \frac{(6+1)\times 6}{2} + \frac{(5+1)\times 5}{2} + \frac{(4+1)\times 4}{2} + \frac{(3+1)\times 3}{2} + 3$$

$$= 27 + 21 + 15 + 10 + 6 + 3 = 82$$

$$P(\bar{E}) = \frac{|\bar{E}|}{|S|} = \frac{8^2}{6^3}$$

Examples

- Drawing 5 cards from a standard deck of 52 poker cards (Four suits: clubs, spades, diamonds, hearts. Each suit has thirteen cards: A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K).
  - What is the probability of the five cards being Royal Flush (i.e., same-suit 10, J, Q, K, A)?

$$|S| = C(52, 5) = \frac{52!}{(52-5)!\,5!}$$

$$\frac{4}{|S|} = \frac{4}{C(52,5)}$$

Examples

- Drawing 5 cards from a standard deck of 52 poker cards (Four suits: clubs, spades, diamonds, hearts. Each suit has thirteen cards: A, 2, 3, …, 10, J, Q, K).
  - What is the probability of the five cards being a **Straight Flush**? Straight flush is a poker hand containing five cards of sequential rank, all of the same suit, such as Q♥ J♥ 10♥ 9♥ 8♥ (a "queen-high straight flush"), but not a royal flush.

$$\frac{(10-1)\times 4}{C(52,5)} = \frac{36}{C(52,5)}$$

Examples

- Drawing 5 cards from a standard deck of 52 poker cards (Four suits: clubs, spades, diamonds, hearts. Each suit has thirteen cards: A, 2, 3, …, 10, J, Q, K).
  - What is the probability of the five cards being a **Four of a Kind**? Four of a kind, also known as **quads**, is a poker hand containing four cards of the same rank and one card of another rank, e.g., 9♣ 9♠ 9♦ 9♥ J♥ ("four of a kind, nines").

$$\frac{13 \times 48}{C(52,5)}$$

Examples

$C(4,3)$ $C(4,2)$

- Drawing 5 cards from a standard deck of 52 poker cards (Four suits: clubs, spades, diamonds, hearts. Each suit has thirteen cards: A, 2, 3, …, 10, J, Q, K).
  - What is the probability of the five cards being a **Full House**? A full house is a poker hand containing three cards of one rank and two cards of another rank, such as 3♣ 3♠ 3♦ 6♣ 6♥.

$$\frac{13 \times C_4^3 \times 12 \times C_4^2}{C(52,5)}$$

Examples

- Drawing 5 cards from a standard deck of 52 poker cards (Four suits: clubs, spades, diamonds, hearts. Each suit has thirteen cards: A, 2, 3, …, 10, J, Q, K).
  - What is the probability of the five cards being a **Flush**? A flush is a poker hand containing five cards all of the same suit, but *not all of sequential rank*, such as K♣ 10♣ 7♣ 6♣ 4♣.

$$\frac{C(13,5) \times 4 - 10 \times 4}{C(52,5)}$$

Examples

- Drawing 5 cards from a standard deck of 52 poker cards (Four suits: clubs, spades, diamonds, hearts. Each suit has thirteen cards: A, 2, 3, …, 10, J, Q, K).
  - What is the probability of the five cards being a **Straight**? A straight is a poker hand containing five cards of sequential rank, *not all of the same suit*, such as 7♣ 6♠ 5♠ 4♥ 3♥.

$$\frac{(4^5 - 4) \times 10}{C(52, 5)}$$