Counting
Principles and Examples

Yan Huang

Product Rule

- If event A can occur in a ways and, *independent of this*, event B can occur in b ways, then the number of ways for both A and B to occur is ab.

For example, drawing two elements, each from an independent set.

$$|A{\times}B| = |A| \cdot |B|$$

$A = \{1, 2, 3\}$

$B = \{\text{'a'}, \text{'b'}\}$

$|\{(1, \text{'a'}), (1, \text{'b'}), (2, \text{'a'}), (2, b), (3, a), (3, b)\}| = 6 = 3 \times 2$

Sum Rule

If event A can occur in a ways and event B can occur in b ways, but *A and B cannot occur together*, then the number of ways for A *or B* to occur is $a + b$.

- E.g., there are 3 daily flights going from Indianapolis to DC and also 2 long-distance bus routes from Indianapolis to DC. How many different ways can you go to DC?
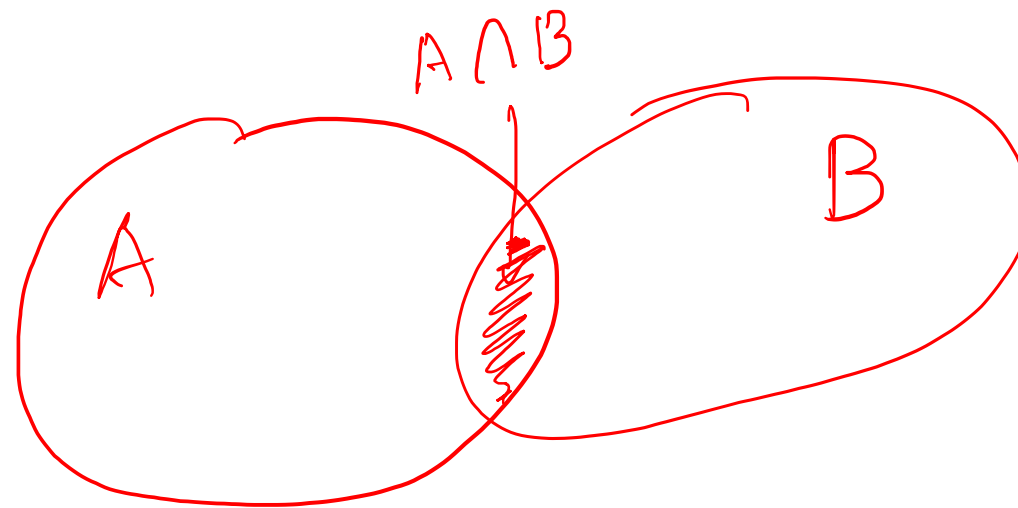
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$a = 3$

$b = 2$

More General Sum Rule

$|A|$: the size of set A.

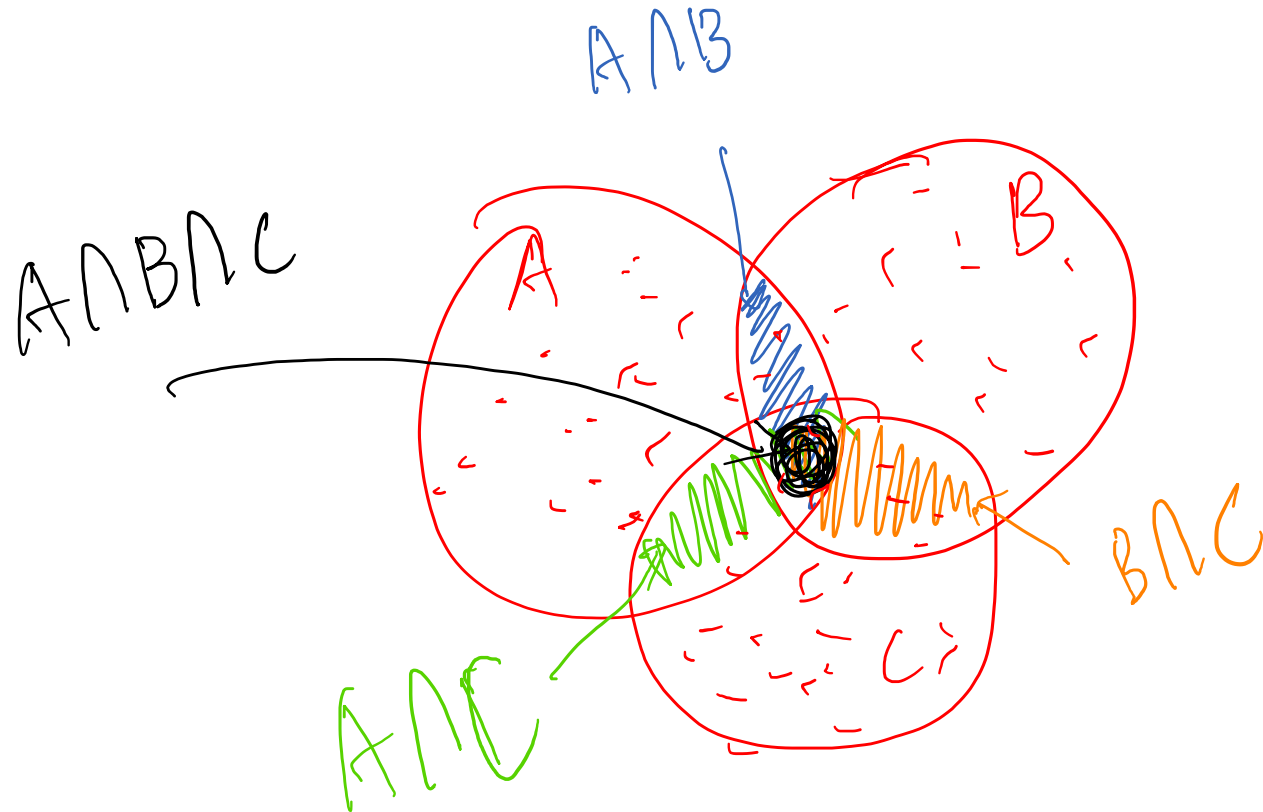If event A can occur in a ways and event B can occur in b ways, but *A and B could occur together*, then

Set disjoint operator

Set join

$$|A \cup B| = |A| + |B| + |A \cap B|$$

$A \cap B$

General Sum Rule for Three Events

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Occupancy Model

Balls and Bins

How many different possible outcomes are there to place k balls into n bins? (All balls and bins are labeled differently.)
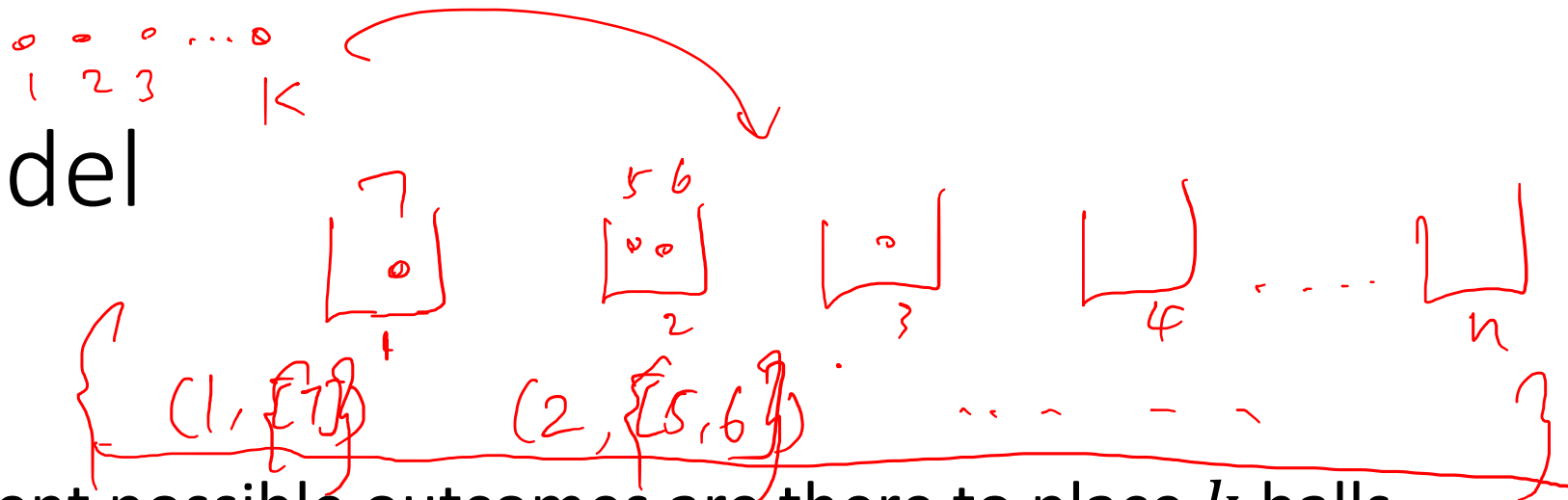
- If a bin can contain any number of balls

if $k=1$, n possible outcomes.

if $k=2$, $n \times n$ possible outcomes.

if $k=3$ $n \times n \times n = n^3$ outcomes.

n^k

$\{(1, \{7\}) \quad (2, \{5,6\}) \quad \cdots \}$

n^k

Occupancy Model

Balls and Bins

How many different possible outcomes are there to place k balls into n bins? (All balls and bins are labeled differently.)

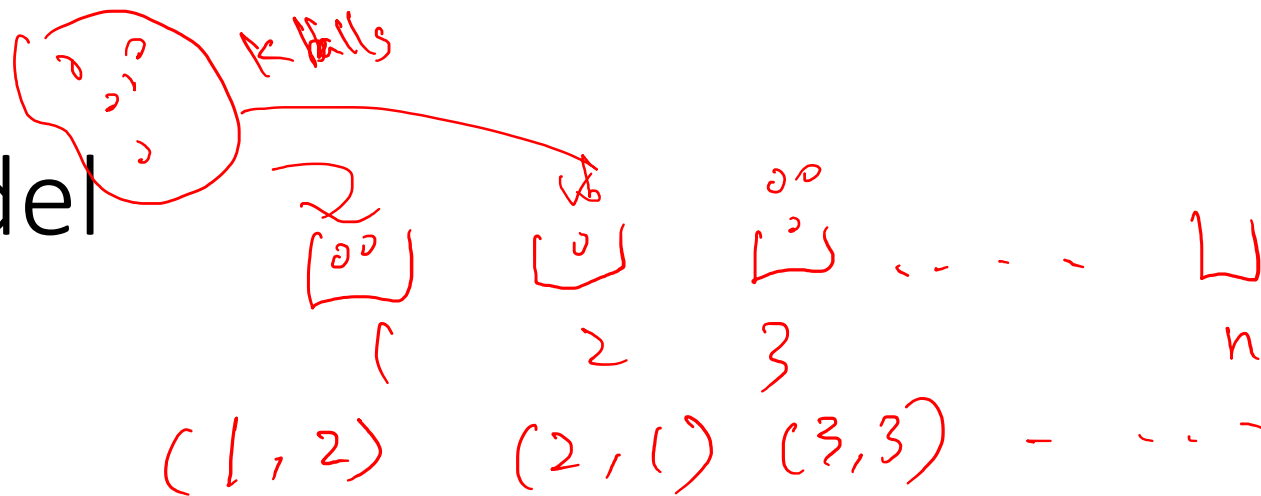- If a bin can contain *at most one ball*

$$P(n, k) = n(n - 1) \cdot \cdots \cdot (n - k + 1) = \frac{n!}{(n - k)!}$$

If $k=1$, n different outcomes.

if $k=2$, $n(n-1)$

$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!} = P(n,k)$

Permutation

Occupancy Model

Balls and Bins

How many different possible outcomes are there to place k balls into n bins? (***All balls are identical*** but all bins are different.)

- If a bin can contain *at most one ball*

$$C(n, k) = \frac{P(n, k)}{k!} = \frac{n!}{k!\,(n - k)!}$$

License plates in Indiana are 3 distinct letters, followed by 3 distinct decimal digits. How many possible license plates are there?
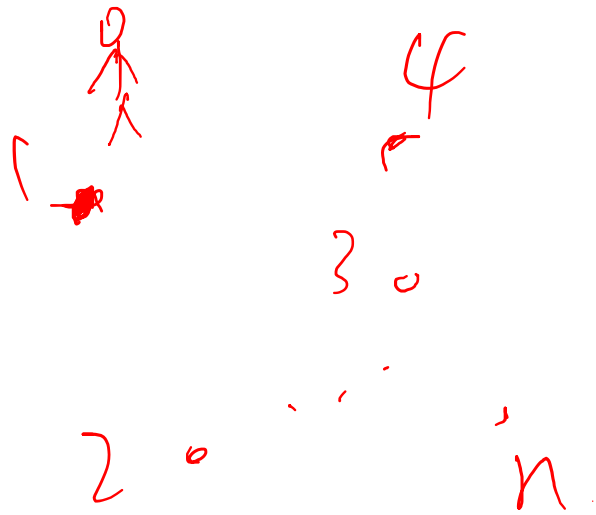
$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 26^3 \times 10^3$

How many ways can we put 3 different balls into 10 different bins? Assume no bin can contain two balls.

or more

$$10 \times 9 \times 8 = P(10,3)$$

How many different ways can a salesman travel among n cities, where he starts in City 1 and visits each other city once and only once?

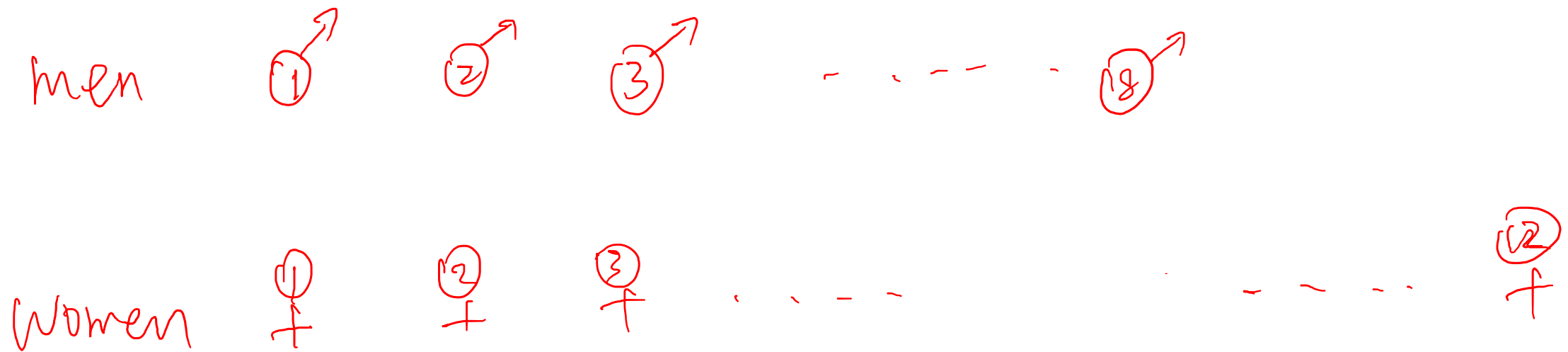$(n-1)(n-2)(n-3) \cdots \times 1$

$= (n-1)!$

How many ways can you select a president, vice president, and treasurer in a club of 30 people?

$$30 \times 29 \times 28 = P(30, 3)$$

How many ways can you form Male-Female dance partners if there are 12 women and 8 men. Assume each man is partnered with some woman (so 4 women go un-partnered).
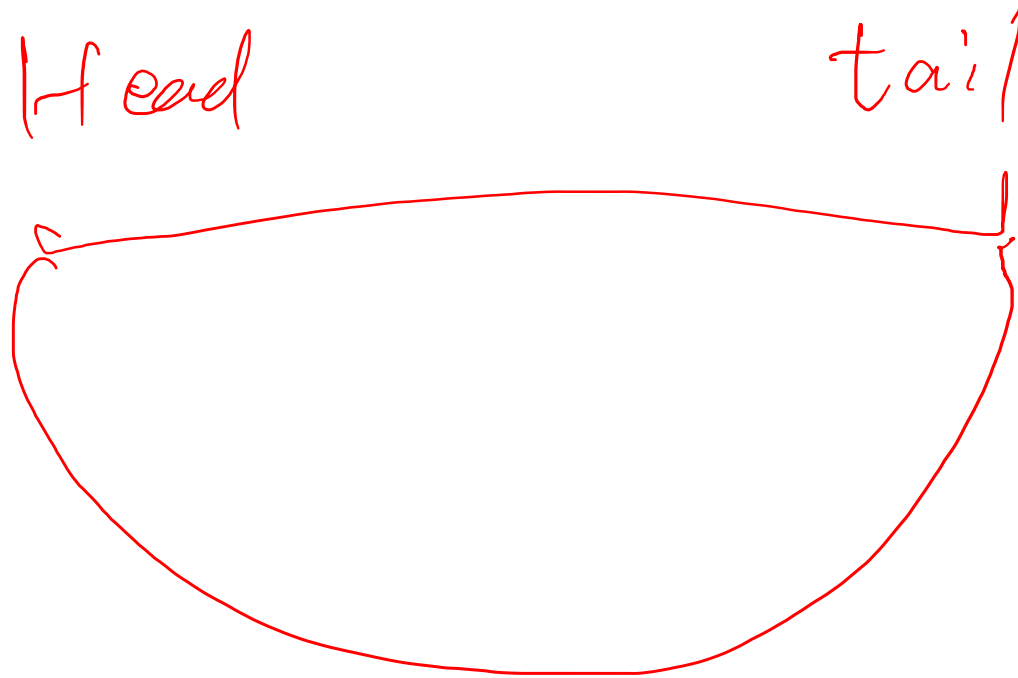
men ♂① ♂② ♂③ - - - - - ♂⑧

women ⚀♀ ⚁♀ ⚂♀ - - - - - - - - ⑫♀

$$P(n, k) = P(12, 8)$$

can

How many ways you position 7 people in a circle?

Head   tail

1 2 3 4 5 6 7
7 1 2 3 4 5 6
6 7 1 2 3 4 5
5 6 7 1 2 3 4
⋮
2 3 4 5 6 7 1

$$\frac{7!}{7} = 6!$$

$\overset{\downarrow}{A} \overset{\downarrow}{B}$

In how many ways can 10 adults and 5 children be positioned in a line so that no two children are next to each other?

Adults: A B C D E F G H I J
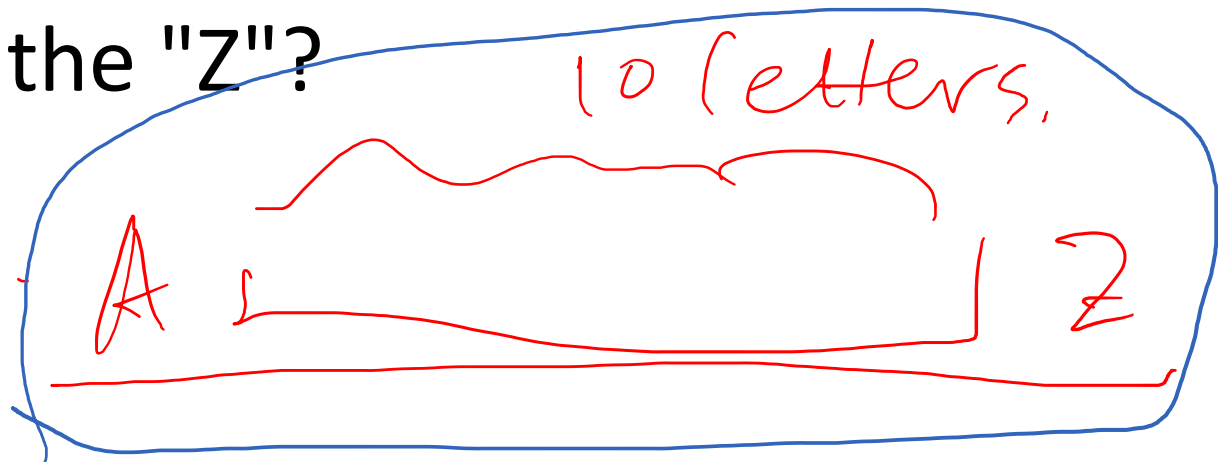
Children: 1 2 3 4 5 $P(10,10) \times P(11,5)$

ex.   A B C 1 D 2 E 3 F G H 4 I J 5   ✓
      A 1 B 2 C 3 D 4 E 5 F G H I J   ✓

How many arrangements are there of the letters A...Z such that there are exactly 10 letters between the "A" and the "Z"?

10 letters.

A $\qquad$ 2

$14! \times P(24,10) \times 15 \times 2$

A group of four people go to a Chinese restaurant that has 100 different dishes. All food will be shared among the four of you. How many ways can you order 4 different dishes?

$$C(100, 4)$$

Tossing a coin 8 times. How many ways can it land with 5 heads total?

H T H H H T H H T

C(8,5)