My research interests lie primarily in the intersection of combinatorics, discrete mathematics, and theoretical computer science.

Most of my work is related to the combinatorics of Coxeter groups, which are certain groups generated by elements of order two. Common examples of Coxeter groups include the dihedral group and the symmetric group (the group of permutations of n elements). Both of these groups have geometric interpretation as the group of symmetries of a regular polygon, and an n-dimensional simplex, respectively. My main contributions in the area can be found in [4, 5, 6, 7, 8]. I explain this work, which is theoretical in nature, in some detail in Section 1.

The projects in which I've been involved more recently have taken a more practical approach as I looked for projects more directly related to computer science. In the last two years, I collaborated on questions relating to the pancake problem and the multi-arm bandit problem. I describe these projects in Section 2 and 3.

In addition, I worked on projects that will not be discussed in detail here that have connections to discrete mathematics and AI. For instance, in [12] we analyze some variations of the classical combinatorial game of Domineering. Moreover, in [1], we give an exact formula for the bandwidth of a graph that comes from the product of paths of the same length. Furthermore, we present some enumerative results counting *Dyck paths*, which can be seen as walks from $(0, 0)$ to (n, n) that lies strictly below the diagonal $y = x$, by two statistics that we define as area and rank of the path. In addition, I have two recent collaborations, one that will be submitted to IJCAI with David Crandall and his Ph.D. students Zhenhua Chen and Chuhua Wang regarding the limitations of AlphaGo to a simple combinatorial game. The second project was recently accepted to AAAI is a project, in collaboration with Larry Moss (IU Math) and CS Ph.D student Caleb Kisby, about a formal logic system to reason about set cardinalities and unions that is complete and decidable in polynomial time. I also have an old project with David Crandall and some of his former students [20] where we tested the ability of computer vision to observe natural events in millions of geo-tagged Flickr photos.

In terms of publications, most mathematicians tend to publish in journals. For my particular field, the *Annals of Combinatorics*, the *Journal of Algebraic Combinatorics*, the *Electronic Journal of Combinatorics*, and *Discrete Applied Mathematics*, are all considered tier "A" journals by a ranking of the Australian Mathematical Society[1] (a ranking usually used by some mathematicians). In terms of CS conferences, one of my collaborations appeared in the ICML proceedings last year, and one was accepted for NeurIPS this year and for AAAI next year.

[1] `https://www.austms.org.au/Rankings/AustMS_final_ranked.html`

1. Coxeter groups and the complete **cd**-index

A *Coxeter group* W can be thought of as a generalization of the dihedral group, the group of symmetries of the n-gon. A Coxeter group is generated by involutions (elements of order two) that satisfy relations that resemble those yielding the rotations of the dihedral group. These groups appear in different branches of mathematics; for instance, they are the Weyl group of semisimple Lie groups and include the symmetric group S_n, the group of reflections of the platonic solids, and the hyperoctahedral group (group of symmetries of an n-dimensional cube).

A *reflection* of W is an element of the form wsw^{-1}, where s is a generator and $w \in W$. A central object in my work on the area, and indeed in the study of Coxeter groups, is the Bruhat graph $B(W)$ of W. This directed graph has the elements of W as vertices and an edge (u, v) if there exists a reflection t such that $ut = v$ and the length of u (the minimum number of generators required to express u) is smaller than the length of v. From this graph, we can define one of the most important partial orders in Coxeter group theory: the Bruhat order $\leq$. We say that $u \leq v$ if there exists a u-v path in the graph $B(W)$. The Bruhat graph $B(u, v)$ corresponding to the interval $[u, v]$ in Bruhat order is the restriction of $B(W)$ to the elements in $[u, v]$.

Dyer [15] defined a total order $<_T$ on the set of reflections of T. One can label the edges of the Bruhat graph using $<_T$, and this labeling can be used to prove nice combinatorial and topological properties of a Bruhat interval $[u, v]$. In fact, $[u, v]$ is the face poset of a regular cell decomposition of a sphere, a result first established by Björner and Wachs [3].

Because $<_T$ is a linear order, one can define the *descent set* of a path Δ in $B(u, v)$ that keeps tracks of the *positions* of the descents of the labels of Δ. For example if the path is labeled $(1, 3, 2, 5, 1)$, then the descent set is $\{2, 4\}$ as $3 > 2$ and $5 > 1$. Billera and Brenti [2] defined a polynomial $\widetilde{\psi}_{u,v}$, called the complete **cd**-index, that encodes the descent sets of u-v paths. I have studied this polynomial in [4, 5, 6]. In particular, in [5], I describe a method of computing $\widetilde{\psi}_{u,v}$ for some Bruhat intervals by looking at an extension of Björner and Wachs' labeling used in [3]. The complete **cd**-index provides an alternative combinatorial description of the *Kazhdan-Lusztig polynomials*, which are relevant in representation theory. I mostly focused on the structure of the shortest u-v paths in $B(u, v)$, which I call the *shortest path poset* of $[u, v]$ and denote it by $SP(u, v)$. I study this poset and provide several combinatorial properties [4, 5, 6, 7]. For example, if there is a unique path in $SP(u, v)$ with empty descent set, then $SP(u, v)$ behaves like a Bruhat interval and the terms corresponding to the complete **cd**-index coming from $SP(u, v)$ form the **cd**-index of $SP(u, v)$ seen as an Eulerian poset.

Two examples of Coxeter groups are the symmetric group and the *group of signed permutations* (which can be seen as the group of symmetries of the n-dimensional cube).

More recent projects have involved questions relating to these groups when being generated not by their Coxeter generators, but by prefix reversals. I describe this in more detail in the next section.

2. PANCAKE GRAPHS

The pancake problem consists of finding the optimal way (in terms of number of operations) to sort a stack of pancakes, each of different size, utilizing only a "chef's spatula". In other words, the only type of operation allowed is to lift pancakes from the top of the stack, flip them using the spatula, and then put the flipped pancakes on top of the stack. The first non-trivial bound was was given by Gates and Papadimitriou [16] (incidentally, this is the only academic paper Bill Gates ever wrote). In general, the pancake problem in NP-hard [14].

It is customary to represent the stack n of pancakes by a permutation on n elements. Now a spatula flip becomes a *prefix reversal*, where one takes a *prefix* of a permutation and reverses it. For instance, 2134, 3214, and 4321 are all the prefix reversals of the permutation 1234. It is customary to denote the i^{th} prefix reversal, which takes the first i characters of a permutation and reverses them, by r_i. There is a graph associated with the pancake problem, called the *pancake graph*, which is defined as follows: For a fixed n, the vertex set is the set of permutations while two permutations u and v are connected by an edge if there is a reversal r_i, with $1 \leq i \leq n$, such that $v = ur_i$. The graph just defined is called the *pancake graph*, and it is denoted by P_n. It is worth noticing that P_n is the *Cayley graph* of the symmetric group generated by prefix reversals. As such, P_n is both *vertex-transitive* and has a low diameter in comparison to the number of vertices of the graph. These properties have made the pancake graph P_n a plausible model for an interconnection scheme for parallel computers [17]. Another interesting feature, established in [17], that makes P_n desirable for parallel computing is the property that P_n contains all cycles of length ℓ, with $6 \leq \ell \leq n!$. Containing all these cycles facilitates local connections within the network.

In recent collaborations [9, 10, 11], we study the cycle structure of the *burnt pancake graph*, BP_n which is the Cayley graph of the group of signed permutations (permutations w of the set $[\pm n] := \{-n, -(n-1), \dots, -1, 1, 2, \dots, n\}$ where $w(i-) = -w(i)$ for all $i \in [\pm n]$). It is customary to represent a signed permutation w by its action on the set $\{1, 2, \dots, n\}$ and simply write $w = w(1)w(2)\cdots w(n)$. In this context, a prefix reversal takes the first i characters of $w(1)w(2)\cdots w(n)$, reverses them and changes their sign. So the prefix reversals of 123 are $(-1)23, (-2)(-1)3$, and $(-3)(-2)(-1)$. So the vertex set of BP_n is the set of signed permutations, and two signed permutations are connected if there is a prefix reversal transforming one into the other. We establish in [10] that BP_n has all cycles of length ℓ, with $8 \leq \ell \leq 2^n n!$, which is more surprising than P_n having a similar property, as BP_n is more sparse than P_n.

We also provide a cycle classification for the 8- and 9-cycles of BP_n in [10, 11] and utilize the classification to count the number of stacks of pancakes that require 4 flips to be sorted. Surprisingly, the count can be made by an elementary polynomial.

Currently we are trying to extend our results to pancakes that have "more sides." Group theoretically, we are now considering the group that is the *wreath product* of a cyclic group and a symmetric group. So if the cyclic group has one element, we recover P_n, and if it has two elements, we recover BP_n. We have been able to establish that a similar behavior is true when considering the cyclic group of three elements. The situation for more than two elements in the cyclic group is more complicated since then the Cayley graphs are directed. The general behavior of the cycle length in more general cases seems to be quite mysterious thus far.

3. Multi-arm Bandit Problem

In my time in the Computer Science department at IU, I have collaborated on machine learning projects relating to the multi-arm bandit problem. In recent collaboration with Yuan Zhou and graduate student Chao Tao, we have looked at variations of the classic *multi-arm bandit problem* from reinforcement learning under different settings [18, 19]. The basic setup is as follows: Suppose one has a set of N slot machines (arms), each of which gives a reward according to an unknown distribution, and one is to come up with an algorithm that will decide which arm (or arms) to select in order to maximize the reward obtained utilizing as few pulls as possible. Sometimes one is given a *fixed budget* T where one is only allowed to pull the arms T times. Our collaboration looks at two settings in particular: (1) a linear setting, where the reward for each arm x (seen as a column vector) is assumed to be of the form $x^T\theta + \epsilon$, where θ is unknown, and (2) using T pulls, we wish to identify which arms have a mean reward higher than a given threshold (this problem is called the *thresholding bandit problem*). Our algorithms from both papers are state-of-the-art and do very well empirically. Furthermore, our ALBA algorithm [19] has an optimal sample complexity (which measures how many pulls are needed to identify the optimal arm with probability close to 1), and our LSA algorithm from [18] is near optimal, up to a double logarithmic term. It is also worth remarking that in the case of the thresholding bandit problem, the standard measure of regret is the probability that at least one of the arms is incorrectly labeled. However, for many applications it is reasonable to try to minimize the *aggregate regret*, the expected number of errors in the classification after T pulls. Indeed, if one is to minimize the simple regret, one could use several pulls trying to classify an arm whose mean reward is close to the threshold and thus ambiguous. So one is better served by moving on from ambiguous arms to correctly classify arms that are less ambiguous.

4. Future Plans

Recently I've been looking at understanding the structure of what we call the *generalized pancake graph*, which is the Cayley graph of the wreath product of C_m, the cyclic group of m elements and S_n is the group of permutations of the set $\{1, 2, \ldots, n\}$. This is a very natural generalization of the pancake graph P_n and the burnt pancake graph BP_n, if $m = 1$ and $m = 2$, respectively. As discussed previously, it was established in [17] that the pancake graph P_n has all cycles from length 6 to $n!$ and we recently established in [10] that BP_n has all cycles from length 8 to $2^n n!$. We have proved that a similar property holds for $m = 3$ when considering the undirected case of the graph. The directed case seems to be a bit more idiosyncratic, and we are trying to understand its structure. Another question that remains unanswered is the utilizing a presentation for the symmetric group and the hyperoctahedral group in terms of prefix reversals to see if such a presentation would help in finding a solution to questions relating to the pancake graph. We have recently found a presentation and are trying to use computational group theory techniques to talk about how long it will take to sort an "average" permutation using only prefix reversals.

References

[1] Louis J. Billera and Saúl A. Blanco. Bandwidth of the product of paths of the same length. *Discrete Appl. Math.*, 161(18):3080–3086, 2013.

[2] Louis J. Billera and Francesco Brenti. Quasisymmetric functions and Kazhdan-Lusztig polynomials. *Israel J. Math.*, 184:317–348, 2011.

[3] Anders Björner and Michelle Wachs. Bruhat order of Coxeter groups and shellability. *Adv. in Math.*, 43(1):87–100, 1982.

[4] Saúl A. Blanco. Shortest path poset of finite Coxeter groups. In *21st International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2009)*, Discrete Math. Theor. Comput. Sci. Proc., AK, pages 189–200. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2009.

[5] Saúl A. Blanco. The complete **cd**-index of dihedral and universal Coxeter groups. *Electron. J. Combin.*, 18(1):Paper 174, 16, 2011.

[6] Saúl A. Blanco. Shortest path poset of Bruhat intervals. *J. Algebraic Combin.*, 38(3):585–596, 2013.

[7] Saúl A. Blanco. Flip posets of Bruhat intervals. *Electron. J. Combin.*, 25(4):Paper 4.16, 16, 2018.

[8] Saúl A. Blanco. Weak Z-property of the absolute order on groups generated by sets closed under taking inverses. *Order*, 36(2):391–397, 2019.

[9] Saúl A. Blanco and Charles Buehrle. Some relations on prefix reversal generators of the symmetric and hyperoctahedral group. *CoRR*, abs/1803.01760, 2018.

[10] Saúl A. Blanco, Charles Buehrle, and Akshay Patidar. Cycles in the burnt pancake graph. *Discrete Applied Mathematics*, 2019.

[11] Saúl A. Blanco, Charles Buehrle, and Akshay Patidar. On the number of pancake stacks requiring 4 flips to be sorted. *CoRR*, abs/1902.04055, 2019.

[12] Saúl A. Blanco and Aviezri S. Fraenkel. Tromping games: tiling with trominoes. *Integers*, 11(3):333–362, 2011.

[13] Saúl A. Blanco and T. Kyle Petersen. Counting Dyck paths by area and rank. *Ann. Comb.*, 18(2):171–197, 2014.

[14] Laurent Bulteau, Guillaume Fertin, and Irena Rusu. Pancake flipping is hard. *Journal of Computer and System Sciences*, 81(8):1556 – 1574, 2015.

[15] Matthew J. Dyer. Hecke algebras and shellings of Bruhat intervals. *Compositio Math.*, 89(1):91–115, 1993.

[16] William H. Gates and Christos H. Papadimitriou. Bounds for sorting by prefix reversal. *Discrete Math.*, 27(1):47–57, 1979.

[17] Arkady Kanevsky and Chao Feng. On the embedding of cycles in pancake graphs. *Parallel Computing*, 21(6):923 – 936, 1995.

[18] Chao Tao, Saúl A. Blanco, Jian Peng, and Yuan Zhou. Thresholding bandit with optimal aggregate regret. *CoRR*, abs/1905.11046, 2019.

[19] Chao Tao, Saúl A. Blanco, and Yuan Zhou. Best arm identification in linear bandits with linear dimension dependency. In *Proceedings of the 35th International Conference on Machine Learning, ICML 2018, Stockholmsmässan, Stockholm, Sweden, July 10-15, 2018*, pages 4884–4893, 2018.

[20] Jingya Wang, Mohammed Korayem, Saul Blanco, and David J. Crandall. Tracking natural events through social media and computer vision. In *Proceedings of the 24th ACM International Conference on Multimedia*, MM '16, pages 1097–1101, New York, NY, USA, 2016. ACM.