Linear Sketches
– A Useful Tool in Streaming and Compressive Sensing

Qin Zhang

Linear sketch

- **Random linear projection** $M : R^n \to R^k$ that preserves properties of any $v \in R^n$ with high prob. where $k \ll n$.

$$\begin{bmatrix} \quad & M & \quad \end{bmatrix} \begin{bmatrix} \\ v \\ \\ \end{bmatrix} = \begin{bmatrix} Mv \end{bmatrix} \longrightarrow \text{answer}$$

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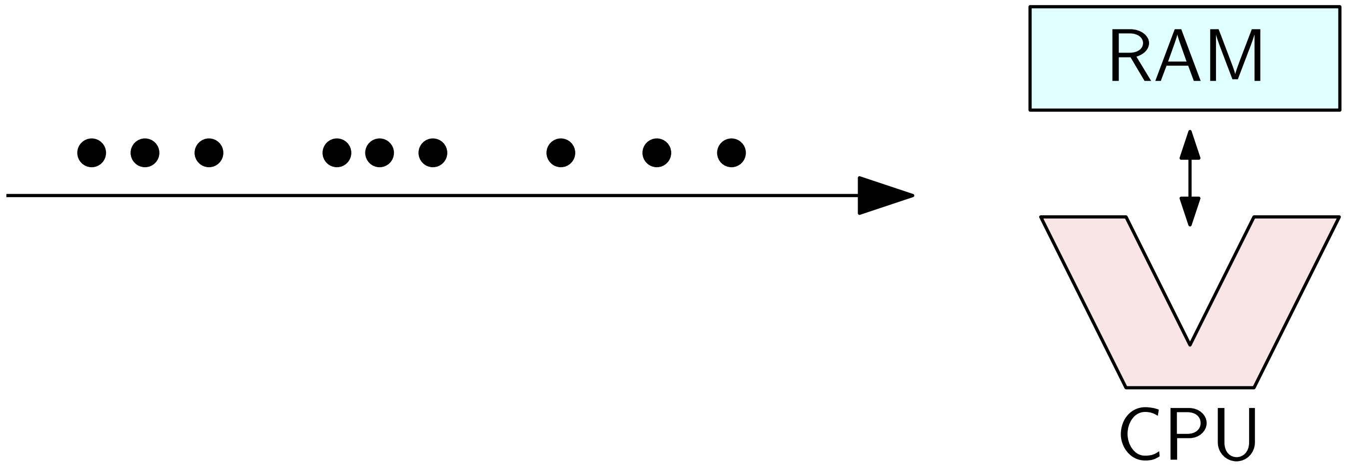
  And rich in theory! You will see in this course.

Data streams

- **The model** (Alon, Matias and Szegedy 1996)



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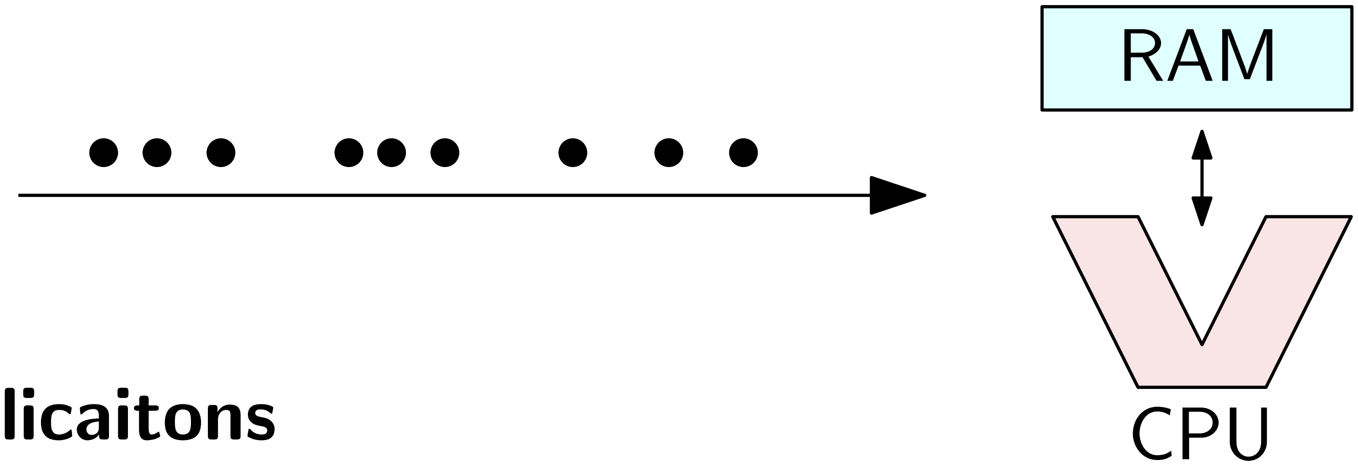

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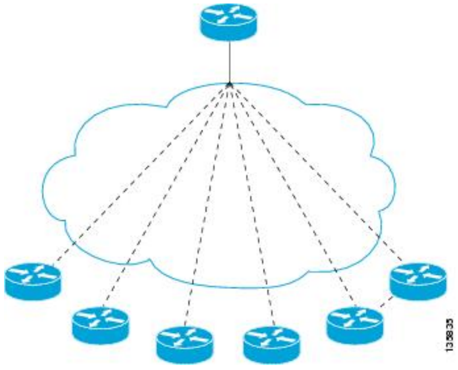

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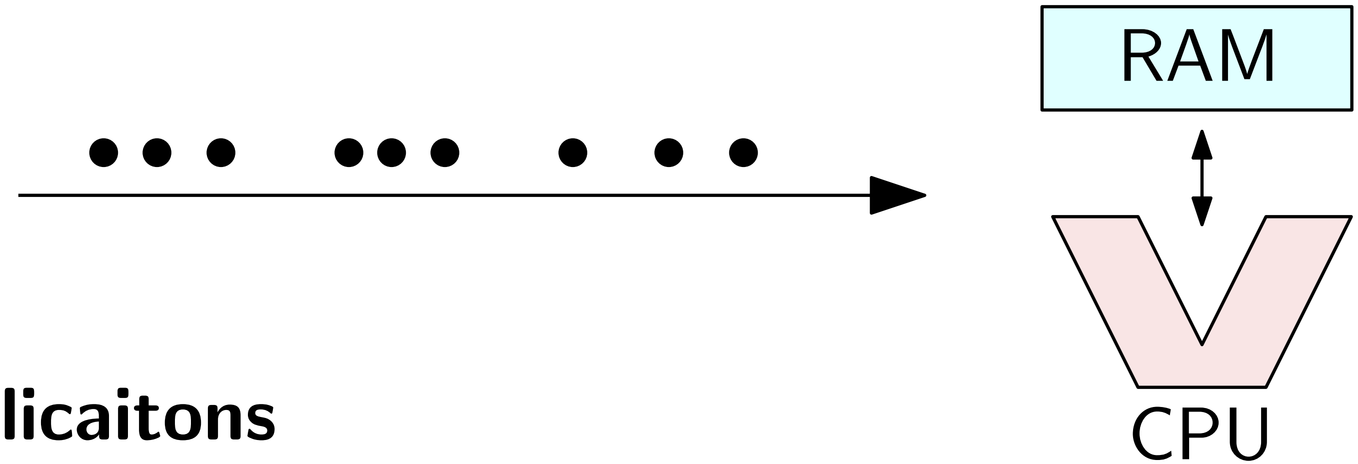

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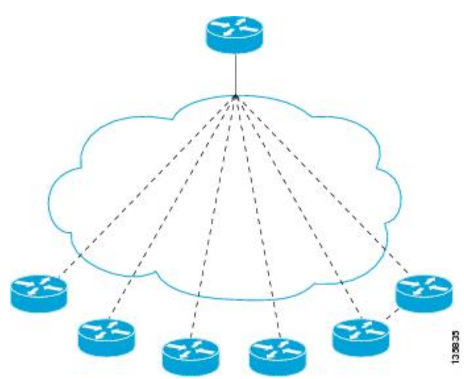

etc.

- **A list of theoretical problems**

Why hard?

- **Game** 1: A sequence of numbers

Why hard?

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52

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- **Game** 1: A sequence of numbers

45

Why hard?

- **Game** 1: A sequence of numbers

18

Why hard?

- **Game** 1: A sequence of numbers

23

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17

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- **Game** 1: A sequence of numbers

41

Why hard?

- **Game** 1: A sequence of numbers

33

Why hard?

- **Game** 1: A sequence of numbers

29

Why hard?

- **Game** 1: A sequence of numbers

49

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35

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Q: What's the median?

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  Alice and Bob become friends

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- **Game** 2: Relationships between Alice, Bob, Carol, Dave, Eva and Paul

  Q: Are Eva and Bob connected by friends?

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- **Game** 2: Relationships between Alice, Bob, Carol, Dave, Eva and Paul

  Q: Are Eva and Bob connected by friends?

  A: YES. Eva $\Leftrightarrow$ Carol $\Leftrightarrow$ Dave $\Leftrightarrow$ Alice $\Leftrightarrow$ Bob

Why hard?

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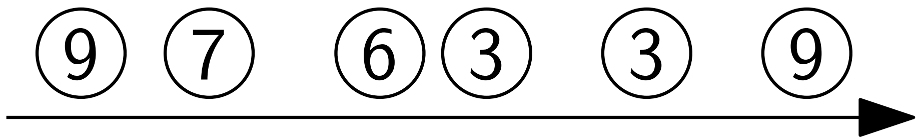
  A: YES. Eva $\Leftrightarrow$ Carol $\Leftrightarrow$ Dave $\Leftrightarrow$ Alice $\Leftrightarrow$ Bob

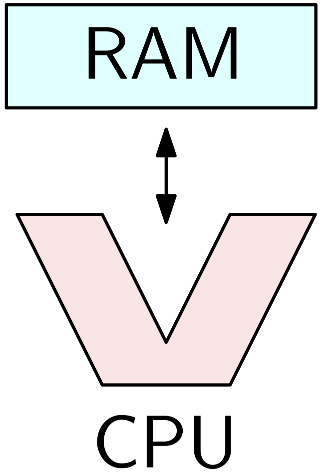
- **Why hard?** Short of memory!

A simple example: distinct elements

- **The problem**



Q: Why linear sketch can be maintained in the streaming model?



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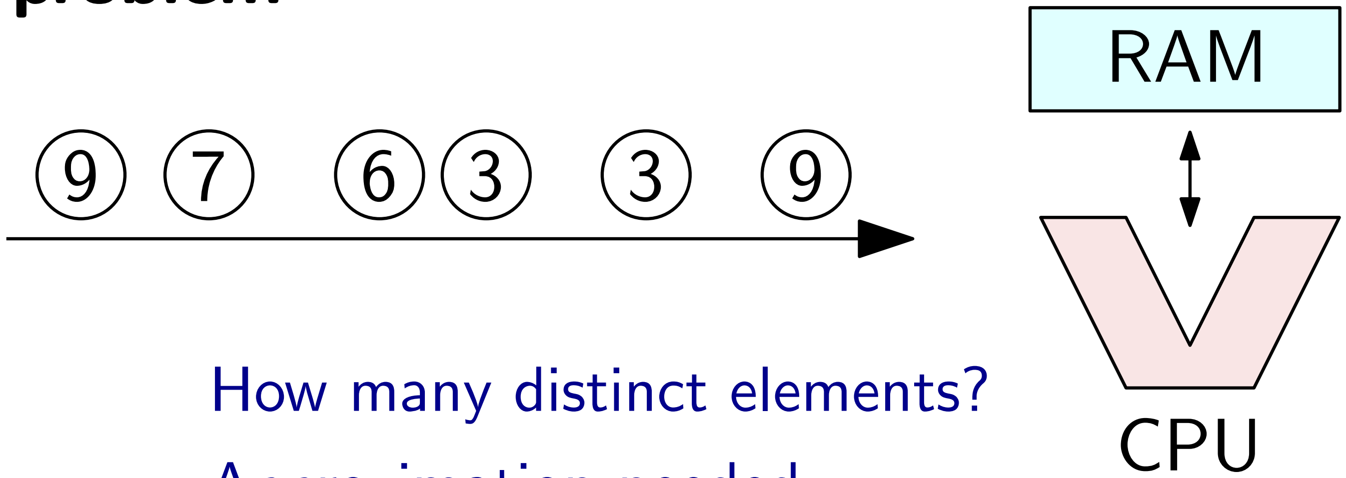

How many distinct elements?

Approximation needed.

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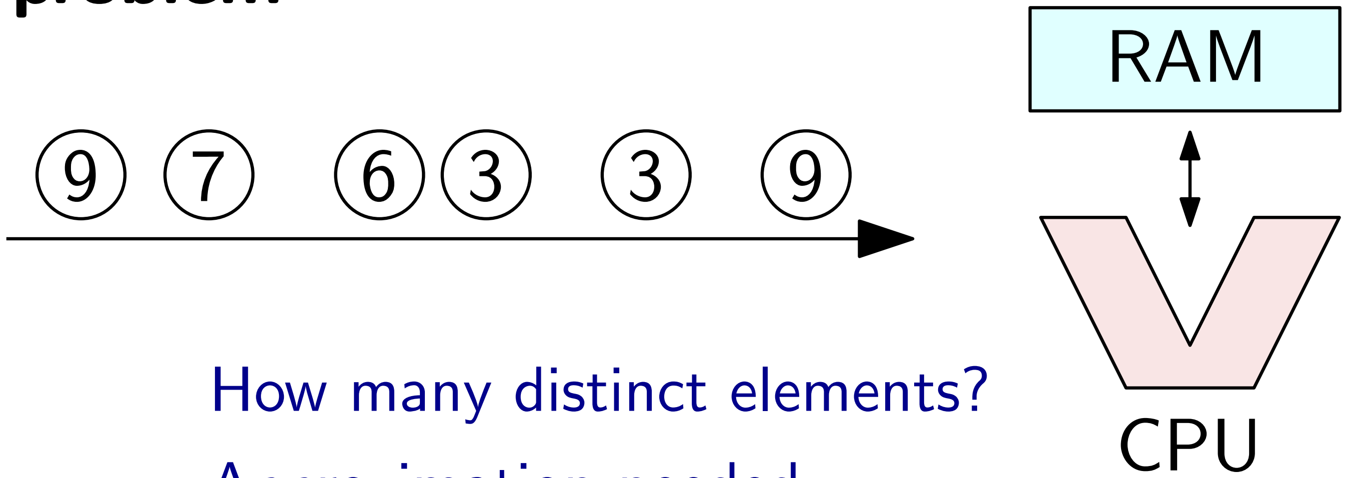

How many distinct elements?

Approximation needed.

- **Search version $\Rightarrow$ Decision version**

Let D be $\#$ distinct elements:

- If $D \geq T(1+\epsilon)$, then answer YES.
- If $D \leq T/(1+\epsilon)$, then answer NO.

Try $T = 1, (1+\epsilon), (1+\epsilon)^2, \ldots$

Now, the decision problem

The algorithm

1. Select a random set $S \subseteq \{1, 2, \ldots, n\}$, s.t. for each i, independently, we have $\Pr[i \in S] = 1/T$
2. Make a pass over the stream, maintaining $Sum_S(x) = \sum_{i \in S} x_i$
   Note: this is a **linear sketch**.
3. If $Sum_S(x) > 0$, return YES, otherwise return NO.

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Lemma

Let $P = Pr[SumS(x) = 0]$. If T is large enough, and ϵ is small enough, then

- If $D \geq T(1 + \epsilon)$, then $P < 1/e - \epsilon/3$.
- If $D \leq T/(1 + \epsilon)$, then $P > 1/e + \epsilon/3$.

(Introduce a few useful probabilistic basics)

Amplify the success probability

Repeat to amplify the success probability

1. Select k sets $S_1, \ldots, S_k$ as in previous algorithm, for $k = C \log(1/\delta)/\epsilon^2$, $C > 0$
2. Let Z be the number of values of $Sum_{S_j}(x)$ that are equal to 0, $j = 1, \ldots, k$.
3. If $Z < k/e$ then report YES, otherwise report NO.

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If the constant C is large enough, then this algorithm reports a correct answer with probability $1 - \delta$.

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Theorem

The number of distinct elements can be $(1 \pm \epsilon)$-approximated with probability $1 - \delta$ using $O(\log n \log(1/\delta)/\epsilon^3)$ words.

Course plan

`www.cse.ust.hk/~qinzhang/HKUST-minicourse/index.html`

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That's all for lecture 1.
Thank you.

Frequency moments and norms

Frequency moments: $F_p = \sum_i |f_i|^p$, f_i: frequency of item i.

- F_0: number of distinct items.
- F_1: total number of items.
- F_2: size of self-join.

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Norms: $L_p = F_p^{1/p}$

L_2 estimation

- **The sketch** for L_2: a linear sketch $Rx = [Z_1, \ldots, Z_k]$, where each entry of $k \times n$ ($k = O(1/\epsilon^2)$) matrix R has distribution $\mathcal{N}(0, 1)$.
  - Each of Z_i is draw from $\mathcal{N}(0, \|x\|_2^2)$.
    Alternatively, $Z_i = \|x\|_2 \, G_i$, where G_i drawn from $\mathcal{N}(0, 1)$.

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- **The estimator**:
  $Y = \text{median}\{|Z_1|, \ldots, |Z_k|\}/\text{median}\{G\}$; $G \sim \mathcal{N}(0,1)$ [a]

[a] M is the median of a random variable R if $\Pr[|R| \leq M] = 1/2$

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- **Sounds like magic?** The intuition behind:
  For "nice"– looking distributions (e.g., the Gaussian), the median of those samples, for large enough # samples, should converge to the median of the distribution.

The proof

- **Closeness in Probability**
  Let $U_1, \ldots, U_k$ be i.i.d. real random variables chosen from any distribution having continuous c.d.f F and median M. Defining $U = \text{median}\{U_1, \ldots, U_k\}$, there is an absolute constant $C > 0$,
  $Pr[F(U) \in (1/2 - \epsilon, 1/2 + \epsilon)] \geq 1 - e^{-Ck\epsilon^2}$

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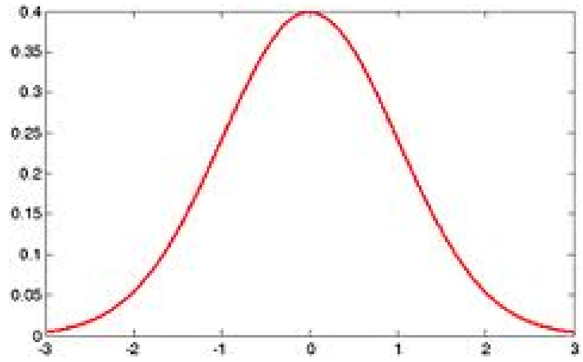
- **Closeness in Value**
  Let F be a c.d.f. of a random variable $|G|$, G drawn from $\mathcal{N}(0, 1)$. There exists an absolute constant $C' > 0$ such that if for any $z \geq 0$ we have $F(z) \in (1/2 - \epsilon, 1/2 + \epsilon)$, then $z = M \pm C'\epsilon$.



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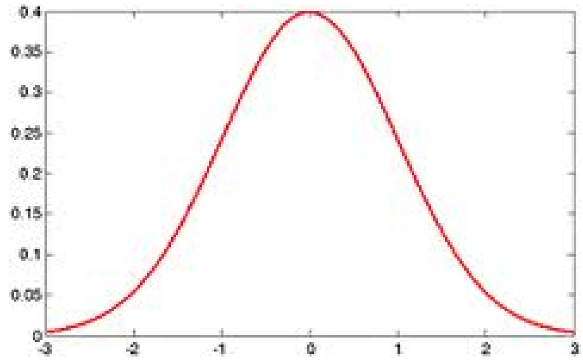
Theorem

$Y = \|x\|_2 (M \pm C'\epsilon)/M = \|x\|_2 (1 \pm C''\epsilon)$, w.h.p.



Generalization

- Key property of **Guassian distribution**:
If $U_1, \ldots, U_n$ and U are i.i.d drawn from Guassian distribution, then $x_1 U_1 + \ldots + x_n U_n \sim \|x\|_p U$ for $p = 2$

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  For $p = 1$, we get **Cauchy distribution** with density function:
  $f(x) = 1/[\pi(1 + x^2)]$

L_p $(p > 2)$ (Not linear mapping but important)

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- First attempt: Use two passes.
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  3. Finally, return $Y = m x_i^{p-1}$.
- Second attempt: Collapse the two passes above
  1. Pick a random element i from the stream, count the number of occurances of i in the rest of the stream, denoted by r.
  2. Now we use r instead of x_i to construct the estimator: $Y' = m(r^p - (r-1)^p)$.

Heavy hitters

- L_p **heavy hitter set**:

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  Given ϕ, ϕ', (often $\phi' = \phi - \epsilon$), return a set S such that

$$HH^p_\phi(x) \subseteq S \subseteq HH^p_{\phi'}(x)$$

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- L_p **Point Query Problem:**
  Given ϵ, after reading the whole stream, given i, report

$$x_i^* = x_i \pm \epsilon \|x\|_p$$

L_2 point query

The algorithm:

[Gilbert, Kotidis, Muthukrishnan and Strauss '01]

- Maintain a sketch Rx such that $s = \|Rx\|_2 = (1 \pm \epsilon)\|x\|_2$ (R is a $O(1/\epsilon^2 \log(1/\delta)) \times n$ matrix, which can be constructed, e.g., by taking each cell to be $\mathcal{N}(0, 1)$)
- Estimator: $x_i^* = (1 - \|Rx/s - Re_i\|_2^2 / 2)s$

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Johnson-Linderstrauss Lemma

$\forall\ x\ \|x\|_2 = \ell$, we have $(1-\epsilon)\ell \leq \|Rx\|_2^2 / k \leq (1+\epsilon)\ell$ w.p. $1-\delta$.

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Theorem

We can solve L_2 point query, with approximation ϵ, and failure probability δ by storing $O(1/\epsilon^2 \log(1/\delta))$ numbers.

L_1 point query

The algorithm for $x \geq 0$ [Cormode and Muthu '05]

- Pick d ($d = \log(1/\delta)$) random hash functions $h_1, \ldots, h_d$ where $h_i : \{1, \ldots, n\} \to \{1, \ldots, w\}$ ($w = 2/\epsilon$).
- Maintain d vectors $Z^1, \ldots, Z^d$ where $Z^t = \{Z^t_1, \ldots, Z^t_w\}$ such that $Z^t_j = \sum_{i:h_t(i)=j} x_i$
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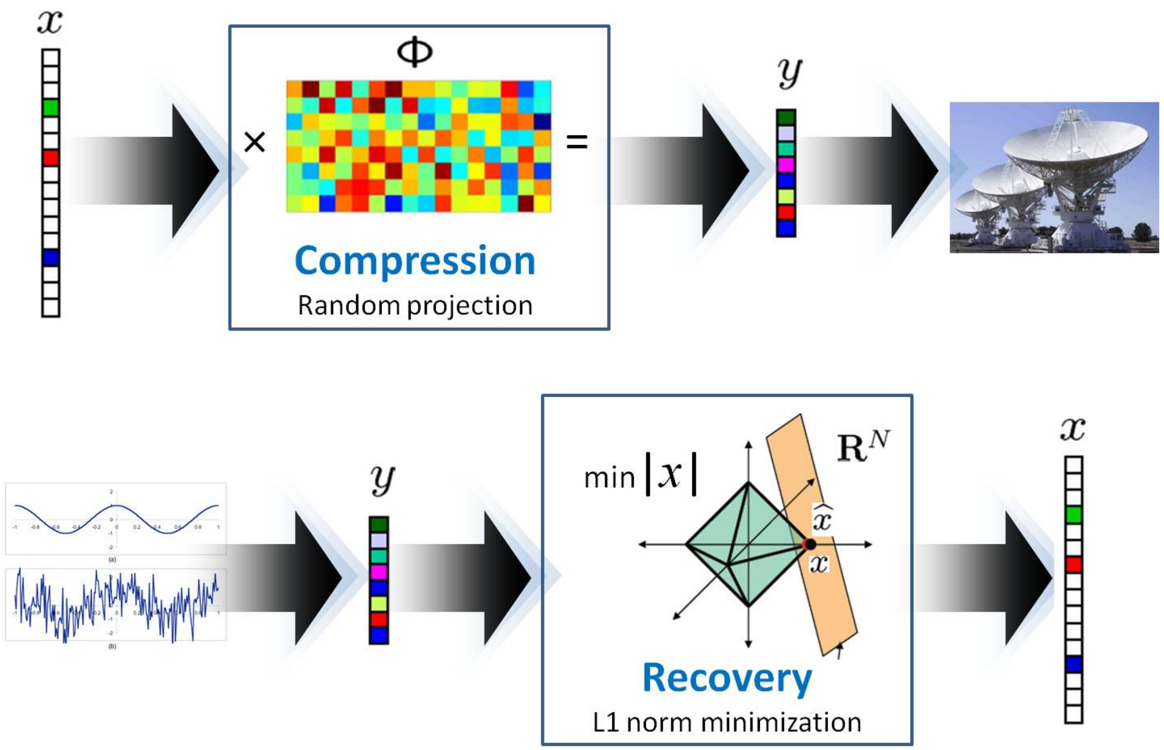
We can solve L_1 point query, with approximation ϵ, and failure probability δ by storing $O(1/\epsilon \log(1/\delta))$ numbers.

Compressive sensing

The model (Candes-Romberg-Tao '04; Donoho '04)



Applicaitons

- Medical imaging reconstruction
- Single-pixel camera
- Compressive sensor network

etc.

Formalization

- *Lp/Lq* **guarantee**: The goal to acquire a signal $x = [x_1, \ldots, x_n]$ (e.g., a digital image). The acquisition proceeds by computing a measurement vector Ax of dimension $m \ll n$. Then, from Ax, we want to recover a k-sparse approximation x' of x so that

$$\|x - x'\|_q \leq C \cdot \min_{\|x''\|_0 \leq k} \|x - x''\|_p \qquad (*)$$

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- Often study: L_1/L_1, L_1/L_2 and L_2/L_2

Formalization

- Lp/Lq **guarantee**: The goal to acquire a signal $x = [x_1, \ldots, x_n]$ (e.g., a digital image). The acquisition proceeds by computing a measurement vector Ax of dimension $m \ll n$. Then, from Ax, we want to recover a k-sparse approximation x' of x so that

$$\|x - x'\|_q \leq C \cdot \underbrace{\min_{\|x''\|_0 \leq k} \|x - x''\|_p}_{Err_p^k(x)} \qquad (*)$$

- Often study: L_1/L_1, L_1/L_2 and L_2/L_2

- **For each**: Given a (random) matrix A, for each signal x, $(*)$ holds w.h.p.

  For all: One matrix A for all signals x. Stronger.

Results

Scale: Excellent | Very Good | Good | Fair

Result Table

Paper	Rand. / Det.	Sketch length	Encode time	Col. sparsity/ Update time	Recovery time	Apprx
[CCF'02], [CM'06]	R	k log n	n log n	log n	n log n	l2 / l2
	R	k $\log^c$ n	n $\log^c$ n	$\log^c$ n	k $\log^c$ n	l2 / l2
[CM'04]	R	k log n	n log n	log n	n log n	l1 / l1
	R	k $\log^c$ n	n $\log^c$ n	$\log^c$ n	k $\log^c$ n	l1 / l1
[CRT'04] [RV'05]	D	k log(n/k)	nk log(n/k)	k log(n/k)	n^c	l2 / l1
	D	k $\log^c$ n	n log n	k $\log^c$ n	n^c	l2 / l1
[GSTV'06] [GSTV'07]	D	k $\log^c$ n	n $\log^c$ n	$\log^c$ n	k $\log^c$ n	l1 / l1
	D	k $\log^c$ n	n $\log^c$ n	k $\log^c$ n	$k^2 \log^c$ n	l2 / l1
[BGIKS'08]	D	k log(n/k)	n log(n/k)	log(n/k)	n^c	l1 / l1
[GLR'08]	D	k $\log n^{\log\log\log n}$	kn^{1-a}	n^{1-a}	n^c	l2 / l1
[NV'07], [DM'08], [NT'08,BD'08]	D	k log(n/k)	nk log(n/k)	k log(n/k)	nk log(n/k) * T	l2 / l1
	D	k $\log^c$ n	n log n	k $\log^c$ n	n log n * T	l2 / l1
[IR'08]	D	k log(n/k)	n log(n/k)	log(n/k)	n log(n/k)	l1 / l1
[BIR'08]	D	k log(n/k)	n log(n/k)	log(n/k)	n log(n/k) *T	l1 / l1
[DIP'09]	D	Ω(k log(n/k))				l1 / l1
[CDD'07]	D	Ω(n)				l2 / l2

Legend:
- n=dimension of x
- m=dimension of Ax
- k=sparsity of x*
- T = #iterations

Approx guarantee:
- l2/l2: $\|x-x^*\|_2 \le C\|x-x'\|_2$
- l2/l1: $\|x-x^*\|_2 \le C\|x-x'\|_1/k^{1/2}$
- l1/l1: $\|x-x^*\|_1 \le C\|x-x'\|_1$

Caveats: (1) only results for general vectors x are shown; (2) all bounds up to O() factors; (3) specific matrix type often matters (Fourier, sparse, etc); (4) ignore universality, explicitness, etc (5) most "dominated" algorithms not shown;

Up to year 2009 ... copied from Indky's talk

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- The algorithm for L_1 point query gives a L_1/L_1 sparse approximation.

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  Recall L_1 Point Query Problem: Given ϵ, after reading the whole stream, given i, report $x_i^* = x_i \pm \epsilon \left\|x\right\|_1$

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  Recall L_1 Point Query Problem: Given ϵ, after reading the whole stream, given i, report $x_i^* = x_i \pm \epsilon \|x\|_1$

- Set $\epsilon = \alpha/k$ and $\delta = 1/n^2$ in L_1 point query. And then return a vector x' consisting of k largest (in magnitude) elements of x^*. It gives w.p. $1 - \delta$,

$$\|x - x'\|_1 \leq (1 + 3\alpha) \cdot Err_1^k$$

  Total measurements: $m = O(k/\alpha \cdot \log n)$

For all (L_1/L_2)

A matrix A satisfies (k, δ)-**RIP (Restricted Isometry Property)** if $\forall$ k-sparse vector x we have $(1-\delta)\|x\|_2 \leq \|Ax\|_2 \leq (1+\delta)\|x\|_2$.

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Johnson-Linderstrauss Lemma

$\forall$ x with $\|x\|_2 = 1$, we have $7/8 \leq \|Ax\|_2 \leq 8/7$ w.p. $1 - e^{-O(m)}$.

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Main Theorem

If A has $(6k, 1/3)$-RIP. Let x^* be the solution to the LP: minimize $\|x^*\|_1$ subject to $Ax^* = Ax$ (x^* is k-sparse). Then

$$\|x - x^*\|_2 \le C/\sqrt{k} \cdot Err_1^k \quad \text{for any } x$$

The lower bounds

- **What's known**: There exists a $m \times n$ matrix A with $m = O(k \log n)$ (can be improved to $m = O(k \log(n/k))$), and a L_1/L_1 recovery algorithm $\mathcal{R}$ so that for each x, $\mathcal{R}(Ax) = x'$ such that w.h.p.

$$\|x - x'\|_1 \leq C \cdot \min_{\|x''\|_0 \leq k} \|x - x''\|_1 .$$

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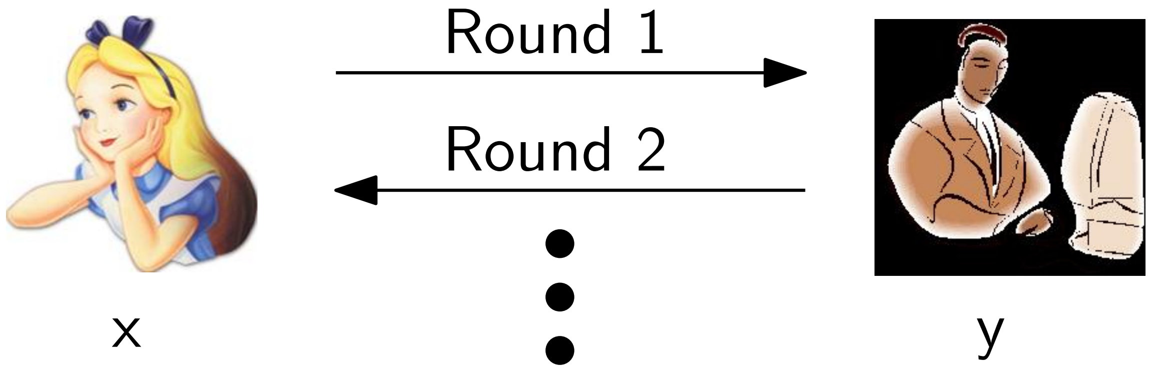
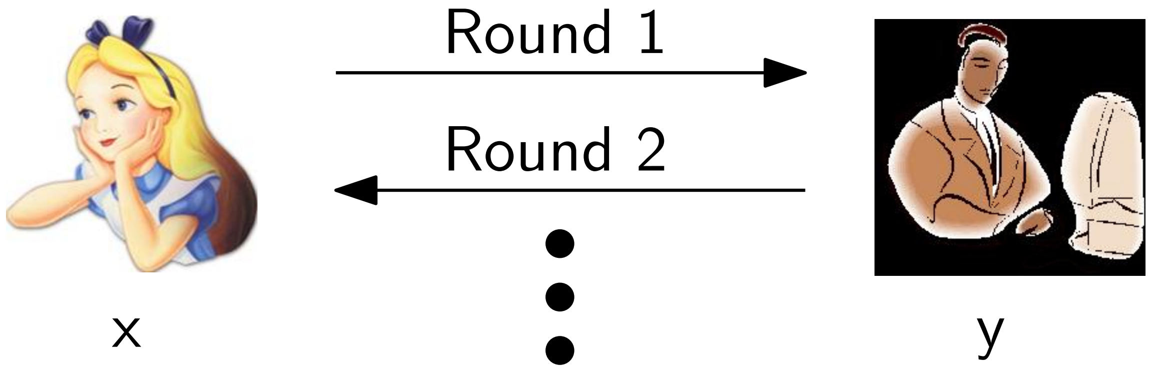
- We are going to show that this is optimal. That is, $m = \Omega(k \log(n/k))$. [Do Ba et. al. SODA '10]
  To show this we need
  - **Communication complexity**
  - **Coding theory**

Communication complexity



They want to jointly compute some function $f(x, y)$

Communication complexity



They want to jointly compute some function $f(x, y)$

We would like to minimize

- Total bits of communication
- # rounds of communication
  (today we only consider 1-round protocol)

Augmented indexing

Promise Input:

Alice gets $x = \{x_1, x_2, \ldots, x_d\} \in \{0, 1\}^d$

Bob gets $y = \{y_1, y_2, \ldots, y_d\} \in \{0, 1, \perp\}^d$
such that for some (unique) i:

1. $y_i \in \{0, 1\}$
2. $y_k = x_k$ for all $k > i$
3. $y_1 = y_2 = \ldots = y_{i-1} = \perp$

Output:

Does $x_i = y_i$ (YES/NO)?

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Theorem

Any 1-round protocol for Augmented-Indexing that succeeds w.p. $1 - \delta$ for some small const δ has communication complexity $\Omega(d)$.

The proof

- Let X be the maximal set of k-sparse n-dimensional binary vectors with minimum Hamming distance k. We have $\log |X| = \Omega(k \log(n/k))$.

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- Restate of the input for Augmented-Indexing (AI):
  Alice is given $y \in \{0,1\}^d$ and Bob is given $i \in d$ and $y_{i+1}, y_{i+2}, \ldots, y_d$. Goal: Bob wants to learn y_i.

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- **A protocol using L_1/L_1 recovery for AI:**
  Set $d = \log|X| \log n$. Let $D = 2C + 3$
  (C is the constant in the sparse recovery)

  Protocol in next slides

The proof (cont.)

A protocol using L_1/L_1 recovery for AI:

1. Alice splits her string y into $\log n$ contiguous chunks $y^1, \ldots, y^{\log n}$, each containing $\log |X|$ bits. She use y^j as an index into X to choose x_j. Alice define: $x = D^1 x_1 + D^2 x_2 + \ldots + D^{\log n} x_{\log n}$.
2. Alice and Bob use shared randomness to choose a random matrix A with orthonormal rows, and round it to A' with $b = O(\log n)$ bits per entry. Alice computes $A'x$ and send to Bob.
3. Bob uses i to compute $j = j(i)$ for which the bit y_i occurs in y^j. Bob also use $y_{i+1}, \ldots, y_d$ to compute $x_{j+1}, \ldots, x_{\log n}$, and he can compute $z = D^{j+1} x_{j+1} + D^{j+2} x_{j+2} + \ldots + D^{\log n} x_{\log n}$.
4. Set $w = x - z$ Bob then computes $A'w$ using $A'z$ and $A'x$
5. From w Bob can recover w' such that $\|w - u - w'\|_1 \le C \cdot \min_{\|x''\|_0 \le k} \|w - u - w''\|_1$.
   where $u \in_R B_1^n(k)$ (the L_1 ball of radius k)
6. From w' he can recover x_j, thus y^j, thus bit y_i

Next topic:

Graph Algorithms

L_0 sampling

Goal: sample an element from the support of $a \in \mathbb{R}^n$

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Algorithm

- Maintain $\tilde{F}_0$, an (1 ± 0.1)-approximation to F_0.
- Hash items using $h_j : [n] \to [0, 2^j - 1]$ for $j \in [\log n]$.
- For each j, maintain:
  - $D_j = (1 \pm 0.1) \left| \{ t \mid h_j(t) = 0 \} \right|$
  - $S_j = \sum_{t, h_j(t)=0} f_t i_t$
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Uniqueness is verified if $D_j = 1 \pm 0.1$. If unique, then $S_j = C_j$ gives identity of the element and C_j is the count.

Graphs

- In semi-streaming, want to process graph defined by edges $e_1, \ldots, e_m$ with $\tilde{O}(n)$ memory and reading sequence in order.

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  To delete e from G: update
  $MA_G \to MA_G - MA_e = MA_{G-e}$,
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  Magic? Mmm, the information of connectivity is $\tilde{O}(n)$:-)

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Lemma

For any subset of nodes $S \subset V$,

$$\text{support}(\textstyle\sum_{i \in S} a_i) = E[S, V \backslash S]$$

Connectivity (cont.)

Sketch: Apply $\log n$ sketches C_i to each a_j

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Run previous algorithm in sketch space::

1. Use $C_1 a_j$ to get incident edge on each node j
2. For $i = 2$ to t:
   - To get an incident edge on supernode $S \subset V$ use:

$$\sum_{j \in S} C_i a_j = C_i(\sum_{j \in S} a_j)$$

Use L_0 sampling algorithm to sample an edge

$$e \in \text{support}(\sum_{i \in S} a_i) = E[S, V \backslash S]$$