# Beyond Group Capacity in Multi-terminal Communications

Mohsen Heidari
EECS Department
University of Michigan
Ann Arbor, USA
Email: mohsenhd@umich.edu

Farhad Shirani
EECS Department
University of Michigan
Ann Arbor, USA
Email: fshirani@umich.edu

S. Sandeep Pradhan
EECS Department
University of Michigan
Ann Arbor, USA
Email: pradhanv@umich.edu

*Abstract*—A new structured coding scheme based on *transversal group* codes is proposed. We investigate the information theoretic performance limits for this strategy in multi-terminal communications. Achievability results are derived for lossless reconstruction of sum of two sources. In addition, a new rate region is presented for the problem of *computation over multiple access channel* . We show that the application of the new coding strategy, results in strict gains in terms of achievable rates in both settings.

## I. INTRODUCTION

Use of structured code ensembles toward obtaining new inner bounds to the optimal performance limits of multi-terminal communication problems has been of recent interest. This approach was started by Korner and Marton [1], who proposed a novel scheme based on linear code ensembles for distributed source coding which outperforms those based on standard unstructured code ensembles. The algebraic structure is exploited toward achieving better source compression. This approach has been applied in a variety of communication settings such as lossy distributed source coding [9], multiple-access channels (MAC) with states [17], interference channels [18], [20], computation over multiple-access channels [8], [10], and broadcast channels [19]. Most of these works have used schemes based on linear code ensembles. All of these works use some form of matching the algebraic structure of the code to the structure of the source or channel.

Codes with weaker algebraic structures such as groups and rings have been studied in [2]-[7] for point-to-point communication problems because of lower computational complexity of encoding and decoding. Recently these codes have been used for multi-terminal communication problems mainly toward achieving better inner bounds to the optimal performance limits that those achieved by linear codes. There are two main reasons to use such codes: 1) the finite fields exist only for alphabets whose size equals to a prime power. 2) there may be a better match between the weaker algebraic structures and the structure of the source or channel. Based on these works, a general framework for *Abelian group codes* which unifies the previous works was provided in [11], and the asymptotic performance limits of group codes for point-to-point channel coding and source coding problems were derived. This was then used for the general distributed source coding problem [9], [15] and a new achievable rate region is derived. The

performance limits of codes over abelian groups which are not fields are, in general, inferior to those of linear codes for the same alphabet size for point-to-point communication problems. However, if the structure of the multi-terminal source or channel matches with that of the former, then they tend to give superior asymptotic performance than those given by the latter. This was demonstrated in [9], [15].

In this paper, a new coding scheme based on *transversal group codes* is proposed. They are based on abelian groups, but they are not closed with respect to the group operation. In fact they are transversals (coset representatives) of group codes. They can also be viewed as nested group codes. We restrict our attention to cyclic groups in this paper. This work is motivated by the construction of multi-level polar codes [5], [21], [22]. We study the information-theoretic performance limits of an ensemble of transversal group codes in point-to-point (ptp) as well as multi-terminal settings. More precisely, four problems are addressed: ptp channel coding, ptp source coding, distributed source coding and computation over MAC. We prove that transversal group codes are optimal in terms of achieving the symmetric capacity of any point-to-point channel and the symmetric rate-distortion function of any source. We use these results as building blocks to study multi-terminal problems. Consider a two-user distributed source coding problem in which the decoder wishes to recover the sum of the sources. We propose a new scheme using transveral group codes and derive an achievable rate region for such problem. The results illustrate that this scheme outperforms all previous schemes such as linear codes and group codes. We also propose a new strategy for a specific linear computation over MAC. In this problem the joint receiver wishes to evaluate the sum of the codewords available at two distributed transmitters. This problem is considered as an intermediate step for other problems such as *broadcast* and *interference* channels. Due to space limitation in this paper, some proofs have been omitted; a more complete version can be found in [23].

This paper is organized as follows: The construction of transveral group codes as well as the preliminaries are presented in Section II. Section III analyzes group codes and transveral group codes for point-to-point settings. Distributed source coding and computation over MAC are addressed in Section IV and V.

## II. Preliminaries

*a) Groups:* A group is a set $\mathbf{G}$ equipped with a binary operation "$+$". All groups considered in this paper are cyclic. A subset $\mathbf{H}$ of $\mathbf{G}$ is a *subgroup*, if it is closed under the group operation. We denote this by $\mathbf{H} \preccurlyeq \mathbf{G}$. For a subgroup $\mathbf{H}$ in $\mathbf{G}$, define a *coset* to be a shift of $\mathbf{H}$ by an element $g \in \mathbf{G}$. A subset $\mathcal{S} \subseteq \mathbf{G}$ is a *transversal* for $\mathbf{H}$ in $\mathbf{G}$ iff every coset of $\mathbf{H}$ contains exactly one element of $\mathcal{S}$. Given a prime power $p^r$, the ring of integers modulo $p^r$, is denoted by $\mathbb{Z}_{p^r} = \{0, 1, \cdots, p^r - 1\}$. For $s = 0, 1, \cdots, r$, define $\mathbf{H}_s = p^s \mathbb{Z}_{p^r} = \{p^s \cdot g : g \in \mathbb{Z}_{p^r}\}$. For any $g \in \mathbb{Z}_{p^r}$, define

$$[g]_s := g + \mathbf{H}_s \qquad (1)$$

*b) Group Codes:* For a group $\mathbf{G}$, let $\mathbf{G}^n = \oplus_{i=1}^n \mathbf{G}$. A group code $\mathcal{C}$ with length $n$ is a subgroup of $\mathbf{G}^\mathbf{n}$. A shifted group code is a translation of $\mathcal{C}$ by an element $b^n \in \mathbf{G}^\mathbf{n}$. The ensemble of group codes is the collection of all subgroups of $\mathbf{G}^n$. Sahebi *et.al.* [11] characterized such ensemble.

*c) Discrete Memoryless Channel:* $(\mathcal{X}, \mathcal{Y}, W_{Y|X})$ denotes a discrete memoryless channel with input alphabet $\mathcal{X}$, output alphabet $\mathcal{Y}$ and conditional probability distribution $W_{Y|X}$. The *Symmetric Channel Capacity* for such channel is defined as $I(X; Y)$, where $X$ is uniform.

*d) Discrete Memoryless Source:* Consider a discrete-time memoryless source that takes values from a set $\mathcal{X}$ with probability distribution $P_X$. The reconstructed source is denoted $Z$ that takes values from $\mathcal{Z}$. The reconstruction is measured by a distortion measure $d : \mathcal{X} \times \mathcal{Z} \to [0, \infty]$. Such a source is denoted by $(\mathcal{X}, \mathcal{Z}, P_X, d)$. The *Symmetric Rate-distortion Function* for such source is defined as

$$R(D) = \min_{P_{Z|X} : \mathbb{E}\{d(X,Z)\} \leq D} I(X; Z)$$

where $Z$ is uniform over $\mathcal{Z}$.

### A. Transversal Group Codes

We propose a construction of *transversal group codes* over the underlying group $\mathbb{Z}_{p^r}$. The proposed construction can be easily extended to any Abelian groups. The coding scheme consists of $r$ codebooks. Each codebook is generated based on a group code over $\mathbb{Z}_{p^r}$. More precisely, given non-negative integers $k_i$, the $i^{th}$ codebook is defined as:

$$\mathcal{C}_i = \{u_i^{k_i} \mathbf{A}_i : u_i^{k_i} \in T_i^{k_i}\} \qquad (2)$$

Where $\mathbf{A}_i$ is a $k_i \times n$ matrix with elements belonging to $\mathbb{Z}_{p^r}$ and $T_i = \{0, 1, \cdots, p^i - 1\}$ is a subset of $\mathbb{Z}_{p^r}$. Consequently, the transversal group code is defined as:

$$\mathcal{C} = \sum_{i=1}^r \mathcal{C}_i + b^n \qquad (3)$$

Where the summation is elementwise and $b^n \in \mathbb{Z}_{p^r}^n$ is a dither vector.

There is an alternative representation for $\mathcal{C}$. Let $\mathcal{J} = \bigotimes_{i=1}^r T_i^{k_i}$. Define the map

$$\phi : \mathcal{J} \to \mathbb{Z}_{p^r}^n \qquad (4)$$

$$\phi(\mathbf{a}) := \sum_{i=1}^r u_i^{k_i} \mathbf{A}_i$$

where $\mathbf{a} = (u_1^{k_1}, u_2^{k_2}, \cdots, u_r^{k_r})$ is an element of $\mathcal{J}$. Therefore, $\mathcal{C}$ is the image of $\phi$ translated by $b^n$. Sometimes, we refer to code-words of $\mathcal{C}$ by $\phi(\mathbf{a}) + b^n$. The resulting rate of the transversal group code is

$$R = \frac{1}{n} \log_2 |\mathcal{J}| = \sum_{i=1}^r \frac{k_i}{n} \log_2 |T_i|$$

$$= \sum_{i=1}^r i \frac{k_i}{n} \log_2 p$$

Given $n$ and $k_i$ for $i = 1, 2, \cdots, r$, the ensemble of transversal group codes is the set of all codes of the form $\mathcal{C}$. Such an ensemble is generated by choosing elements of $\mathbf{A}_i$ and $b^n$ uniformly and independently from $\mathbb{Z}_{p^r}$.

Note, based on (2), only $\mathcal{C}_r$ is a group code. Other $\mathcal{C}_i$'s are a subset of codewords of a group code. Therefore, a transversal group code has a weaker algebraic structure than a group code. We will see later that this strategy is more flexible terms of matching the structure of certain sources and channels. Therefore, transversal group codes can be superior to group codes. We first present the results for point-to-point problems

## III. Point-to-point Settings

### A. Channel Coding

Let $(\mathcal{X}, \mathcal{Y}, W_{Y|X})$ be a discrete memoryless channel with $\mathcal{X} = \mathbb{Z}_{p^r}$. We present the capacity of the ensemble of group codes as derived in [3].

**Theorem 1** ([3])**.** *For the class of symmetric channels $(\mathcal{X} = \mathbb{Z}_{p^r}, \mathcal{Y}, W_{Y|X})$, the group code capacity is given by:*

$$C_g = \min_{0 \leq s \leq r-1} \frac{r}{r-s} I(X; Y | [X]_s)$$

*where $X$ is uniform over $\mathbb{Z}_{p^r}$.*

Theorem 1 shows the drawback of using group codes. Fix $s$, consider a channel in which $Y$ depends highly on $[X]_s$. In this case, $\frac{r}{r-s} I(X; Y | [X]_s)$ is small causes a rate-loss. One solution is to add another codebook to compensate for this rate-loss. This is the reason that we use a summation of different codebooks in transversal group codes. Example 1 illustrates the effects of adding another codebook to the original group code.

For the channel $(\mathcal{X} = \mathbb{Z}_{p^r}, \mathcal{Y}, W_{Y|X})$, we use a random coding argument using transversal group codes over the underlying group $\mathbb{Z}_{p^r}$. The random encoder is characterized by a random map $\phi$ and random shift $B^n$. Given a message $\mathbf{a} \in \mathcal{J}$, the encoder sends $x^n = \phi(\mathbf{a}) + B^n$. Suppose $P_X$ is a uniform probability distribution over $\mathbb{Z}_{p^r}$. Upon receiving $y^n$, the decoder looks for $\tilde{\mathbf{a}} \in \mathcal{J}$ such that $\phi(\tilde{\mathbf{a}}) + B^n$ is jointly

typical with $y^n$. The decoder declares error if there is no such $\tilde{\mathbf{a}}$ or if $\tilde{\mathbf{a}}$ is not unique. We assume the message sequence $\mathbf{a}$ is chosen uniformly from $\mathcal{J}$.

**Theorem 2.** *For channels with input alphabet $\mathcal{X} = \mathbb{Z}_{p^r}$, transversal group codes achieve the symmetric capacity.*

    *Proof:* Omitted. ■

Next we compare transversal group codes and group codes on $\mathbb{Z}_4$.

**Example 1.** Consider the channel $(\mathcal{X} = \mathbb{Z}_4, \mathcal{Y} = \{0, 1\}, W)$ where

$$W(0|x) = \mathbb{1}\{x \in \{0, 2\}\}$$
$$W(1|x) = \mathbb{1}\{x \in \{1, 3\}\}.$$

By Theorem 1 the capacity of group codes for this channel is

$$\min\{I(X; Y), 2I(X; Y|[X])\} \tag{5}$$

Therefore, the capacity of group codes in this case is 0. However, by Theorem 2, transversal group codes achieve $I(X; Y) = 1$.

### B. The Source Coding Problem

Let $(\mathcal{X}, \mathcal{Z} = \mathbb{Z}_{p^r}, P_X, d)$ be a discrete and memoryless source. The problem of source coding using group codes is studied in [11]. An achievable rate for such codes is given in the following theorem.

**Theorem 3** ([11])**.** *Suppose $Z$ is a uniform random variable over $\mathbb{Z}_{p^r}$. For a given a distortion $D$, The following rates are achievable using group codes for the source $(\mathcal{X}, \mathcal{Z} = \mathbb{Z}_{p^r}, P_X, d)$:*

$$R \geq \min_{P_{Z|X}: \mathbb{E}\{d(Z, X)\} \leq D} \max_{1 \leq s \leq r} \frac{r}{s} I([Z]_s; X).$$

Consider the ensemble of transversal group codes defined in Section II-A. We use this ensemble to achieve the symmetric rate-distortion function:

**Theorem 4.** *Transversal group codes achieve the symmetric rate-distortion function for the source $(\mathcal{X}, \mathcal{Z} = \mathbb{Z}_{p^r}, P_X, d)$.*

    *Proof:* Omitted. ■

By a similar argument as in the channel coding problem, one can show that transversal group codes outperform group codes.

## IV. Distributed Source Coding

In the two-user distributed source coding problem, each encoder observes a source sequence. The sources are correlated. Each encoder sends a quantized version of its corresponding source sequence to the central decoder. The decoder wishes to reconstruct a function of the two sources within some distortion level. This problem is well studied for several special cases [12], [13], [14], [9], [15].

**Definition 1** (Distributed Sources)**.** *Consider a pair of sources with joint distribution $P_{XY}$ defined on $\mathcal{X} \times \mathcal{Y}$. This set-up is denoted as $(\mathcal{X}, \mathcal{Y}, P_{XY}, \rho)$. The source sequences $(X^n, Y^n)$ are generated randomly and independently with the joint distribution $P(x^n, y^n) = \prod_{i=1}^{n} P_{XY}(x_i, y_i)$. Let $\mathcal{Z}$ be a discrete set and the function $\rho$ be given by $\rho : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$.*

An $(n, M_1, M_2)$-code involves two encoder maps

$$f_1 : \mathcal{X}^n \to \{1, 2, \cdots, M_1\}$$
$$f_2 : \mathcal{Y}^n \to \{1, 2, \cdots, M_2\}$$

and a decoder

$$g : \{1, 2, \cdots, M_1\} \times \{1, 2, \cdots, M_2\} \to \mathcal{Z}^n$$

The pair $(R_1, R_2)$ is said to be achievable if for every $\epsilon > 0$ and large enough $n$, there exists an $(n, M_1, M_2)$-code such that

$$\frac{1}{n} \log_2 M_i \leq R_i + \epsilon \quad for \quad i = 1, 2$$

and

$$P\{\rho(X^n, Y^n) \neq g(f_1(X^n), f_2(Y^n))\} \leq \epsilon.$$

The special case where $\mathcal{X} = \mathcal{Y} = \mathbf{G}$ where $\mathbf{G}$ is a group and $\rho(X, Y) = X + Y$, is well studied in [9], [15]. A new coding scheme based on group codes is introduced in [15] and an achievable rate region is derived. The results indicate that using a combination of group codes and unstructured random codes one can achieve rates outside of the previously known rate regions. For ease of exposition, we only concentrate on the special case where $\mathbf{G} = \mathbb{Z}_{p^r}$. This can be easily generalized to any Abelian groups. For simplicity, we denote such distributed sources over the group $\mathbf{G}$ by $(\mathbf{G}, P_{XY})$. We first present the achievable rate region using group codes [15].

Suppose $X$ is a random variable taking values from $\mathbb{Z}_{p^r}$. For $0 \leq \theta \leq r - 1$, define

$$H_\theta(X) = \frac{r}{r - \theta} H(X|[X]_\theta)$$

For $\theta = 1$, let $H_r(X) = r$. Note $H_0(X) = H(X)$.

**Theorem 5** ([15])**.** *Let $X, Y$ be two sources with input alphabets $\mathcal{X} = \mathcal{Y} = \mathbb{Z}_{p^r}$ and joint distribution $P_{XY}$. For the lossless reconstruction of the sum $Z = X + Y$, the following rates are achievable using group codes:*

$$R_1 = R_2 \geq \max_{0 \leq \theta \leq r - 1} H_\theta(Z) \tag{6}$$

**Theorem 6.** *In the previous theorem, the following rates are achievable using transversal group codes:*

$$R_1 = R_2 \geq H(Z) + \sum_{s=1}^{r} \beta_s \big| H_s(Z) - \max_{0 \leq \theta \leq s-1} H_\theta(Z) \big|^+ \tag{7}$$

*where*

$$\beta_s = \frac{H(V_s|[V_s]_s)}{s \log p},$$

*and $V_s$ is a uniform random variable over $\{0, 1, \cdots, p^s - 1\}$.*

*Proof:* Omitted. ∎

The bound in (7) contains (6).

## V. Computation Over MAC

Consider the problem in which a central receiver is interested in reconstructng the sum of two codewords. Suppose $X_1$ and $X_2$ are two random variables taking values from a group $\mathbf{G}$. Two distributed encoders send $X_1, X_2$ through a MAC. The decoder wishes to decode $X_1 + X_2$. Fig. 1 depicts the problem. The MAC is defined by the conditional probability $W_{Y|X_1,X_2}$, where $Y$ is the channel's output with alphabet $\mathcal{Y}$. This set-up is denoted by $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}, W)$.
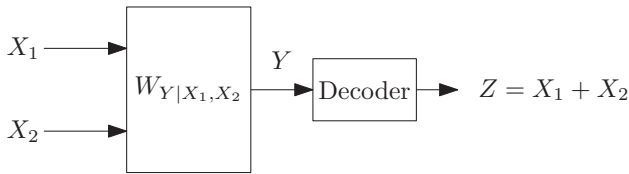


Fig. 1. An example of computation over MAC.

**Definition 2.** *(Codes for computation over MAC) Consider positive integers $n, k$. A $(k,n)$-code for computation over MAC consists of two encoding functions and one decoding function. The encoding function is given by $f_i : \mathcal{S}_i^k \to \mathcal{X}^n$ for $i = 1, 2$ and the decoding function is defined by the map $g : \mathcal{Y}^n \to \mathcal{X}^k$.*

**Definition 3.** *(Achievable rate) We say $R = \frac{k}{n}$ is achievable if for any $\delta > 0$, there exist a $(k,n)$-code such that*

$$P\{g(Y^n) \neq X_1^k + X_2^k\} \leq \delta$$

A study of computation over MAC for the case where $\mathcal{X}$ is a Galois field, can be found in [1], [10], [16]. Nazer and Gastpar [10] used linear codes to generalize the work of Korner and Marton [1] to $q$-ary alphabet. Coset codes are employed to further extend the results to an arbitrary MAC [16]. In this section, we generalize the results when $\mathcal{X} = \mathbb{Z}_{p^r}$. We first derive an achievable rate region using group codes. Then, we employ transversal group codes to improve upon the previously known rate regions.

**Theorem 7.** *For computation over MAC where the input alphabets are $\mathbb{Z}_{p^r}$, group codes achieve the following bound:*

$$R_{g,mac} \leq \min_{0 \leq s \leq r-1} \frac{r}{r-s} I(X_1 + X_2; Y | [X_1 + X_2]_s)$$

*where $X_1$ and $X_2$ are distributed uniformly and are independent of each other.*

*Proof:* Omitted. ∎

We propose a new scheme which involves an inner and an outer code. The inner code is a modified version of the transversal group code defined in (2). In this code, instead of $T_i$, elements of $u_i^{k_i}$ belong to the whole group $\mathbb{Z}_{p^r}$. Therefore, we obtain a larger codebook than $\mathcal{C}$ in (3). We use a particular binning to reduce the size of this codebook.

*Binning:* Contrary to the common coding schemes, binning is used only at the decoder. Binning is different for each layer of the code. For the $i$th layer, two codewords $u_i^{k_i} \mathbf{A}_i$ and $\tilde{u}_i^{k_i} \mathbf{A}_i$ belong to the same bin if $u_i^{k_i} - \tilde{u}_i^{k_i}$ belongs to the subgroup $\mathbf{H}_i^{k_i}$. Therefore, cosets of $\mathbf{H}_i^{k_i}$ determine the binning for the $i$th layer. Observe that elements of $T_i$ can be used to index each bin.

Identical encoders are used for each terminal. In the encoders we use $\phi$ in (4) with $\mathcal{J}$ replaced by $\mathbf{K} = \bigoplus_{i=1}^r \mathbb{Z}_{p^r}^{k_i}$. Suppose $\mathbf{a}_1$ and $\mathbf{a}_2$ are sent. Let $\mathbf{z} = \mathbf{a}_1 + \mathbf{a}_2$. Upon receiving $y^n$, the decoder wishes to recover the bin number associated with $\mathbf{a}_1 + \mathbf{a}_2$. Therefore, it first finds $\tilde{\mathbf{z}} \in \mathbf{K}$ such that $\phi(\tilde{\mathbf{z}}) + b_1^n + b_2^n$ is jointly typical with $y^n$ with respect to the distribution $P_Z \cdot W_{Y|X_1+X_2}$ where $P_Z$ is the uniform distribution over $\mathbb{Z}_{p^r}$. Then it outputs the bin number of $\tilde{\mathbf{z}}$.

Suppose there is no error in recovering the inner code. We use different outer codes in each layer. Given there are no errors in recovering the previous layers, the outer code for the $i$th layer sees the MAC $(\mathcal{S} = T_i, \mathcal{V}_i, Q_{V_i|S_1+S_2})$, where

$$Q_{V_i|S_1+S_2} = \mathbb{1}\{V_i = [S_1 + S_2]_i\}$$

Therefore, this outer code should a code for computation over MAC for this channel.

**Lemma 1.** *An achievable rate for computation over the MAC $(\mathcal{S} = T_i, \mathcal{V}_i, Q_{V_i|S_1+S_2})$ is*

$$C_i = i \log_2 p - H(S_1 + S_2 | [S_1 + S_2]_i)$$

*where the addition is the group $\mathbb{Z}_{p^r}$ operation and $S_1, S_2$ are uniform random variables taking values from $T_i$.*

For $0 \leq \theta \leq r - 1$, define $I_\theta(X; Y) = \frac{r}{r-\theta} I(X; Y | [X]_\theta)$. Also let $I_r(X; Y) = 0$. We derive the achievable rate region for this new scheme.

**Theorem 8.** *An achievable rate using transversal group codes for the MAC channel $(\mathcal{X} = \mathbb{Z}_{p^r}, \mathcal{Y}, W_{Y|X_1+X_2})$ is*

$$R \leq \sum_{s=1}^r \alpha_s \Big| \min_{0 \leq \theta \leq s-1} I_{[p^\theta]} - I_{[p^s]} \Big|^+$$

*where $\alpha_s = \frac{C_s}{s \log p}$.*

*Proof:* Omitted. ∎

**Example 2.** Consider the MAC depicted in Fig. 2. Suppose the inputs take values from $\mathbb{Z}_4$. $X_1 + X_2$ is passed through the channel $(\mathcal{Z} = \mathbb{Z}_4, \mathcal{Y}, W_{Y|Z})$. $W_{Y|Z}$ is given in Fig. 3, where $0 \leq \epsilon \leq 1$. The receiver wishes to decode $\tilde{X}_1 + X_2$.

The point to point capacity of this channel is 1. Suppose $Z$ is uniform over $\mathbb{Z}_4$, then

$$I(Z; Y) = h\left(\frac{1+\epsilon}{4}\right) - \frac{h(\epsilon)}{4} \tag{8}$$

$$I(Z; Y | [Z]) = 0.5 h\left(\frac{1-\epsilon}{2}\right) - \frac{1}{4} h(\epsilon). \tag{9}$$

For this problem, the best achievable rate using group codes is given by

$$R_1 = R_2 = \min\{I(Z;Y), 2I(Z;Y|[X_1 + X_2])\}, \quad (10)$$

whereas transversal group codes achieve:

$$R_1 = R_2 = 0.5I(Z;Y) + I(Z;Y|[X_1 + X_2]). \quad (11)$$
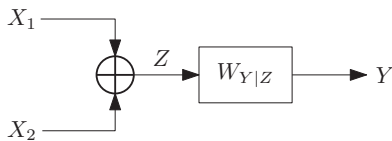
Fig. 4 depicts the achievable sum-rates for different $\epsilon$.

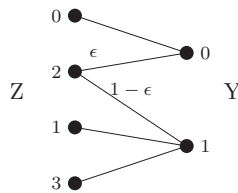Fig. 2.   The multiple access channel in Example 2

Fig. 3.   The channel between $Z$ and $Y$.

## VI.   CONCLUSION

New coding strategies based on transversal group codes are proposed for multi-terminal communications. Achievablility results are provided for the problem of computation over MAC as well as distributed source coding. It is shown that our coding schemes outperform all previously known strategies for these communication problems.

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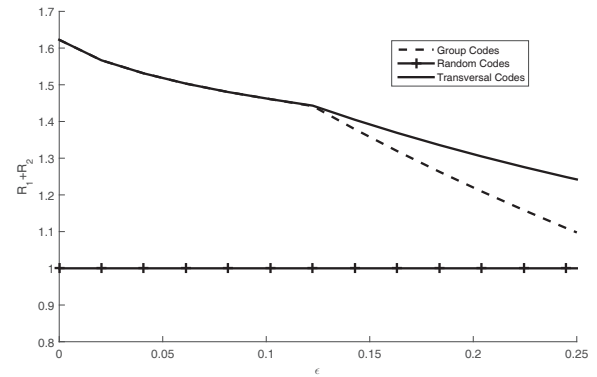
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Fig. 4.   Acheivable Sum-rate for example 2