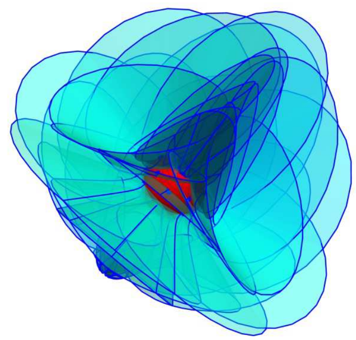

Visualizing the Eguchi-Hanson Space

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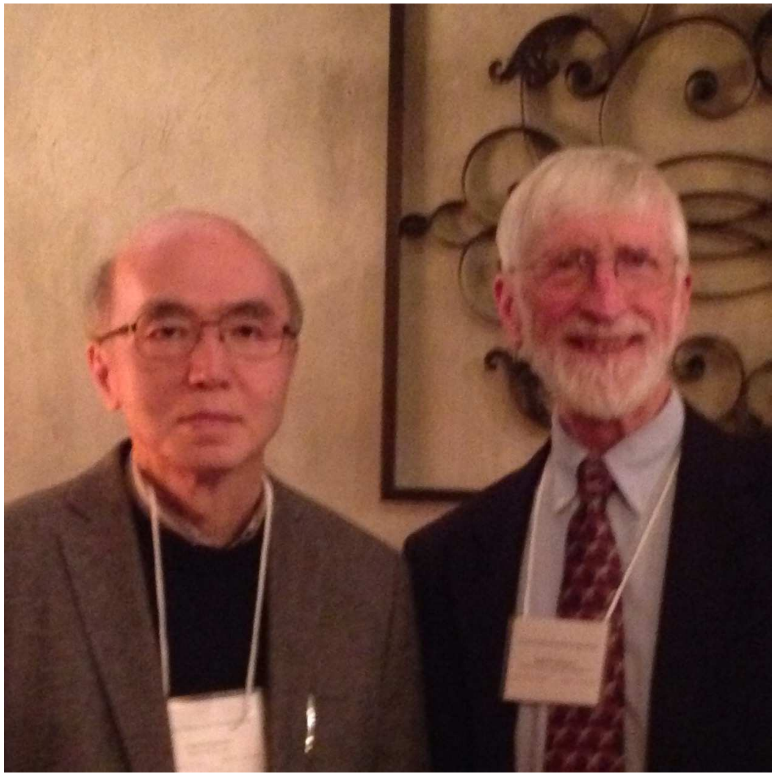
Visualization Workshop, Komaba Campus Univ. of Tokyo

14–15 March 2020

Some background . . .

I am here for the **Tohru Eguchi Memorial Workshop** held last week at the Tokyo Kashiwa Campus, remembering Eguchi and our work together as postdocs in 1978 on the Eguchi-Hanson metric.



Nambu Memorial Symposium
University of Chicago, March 2016

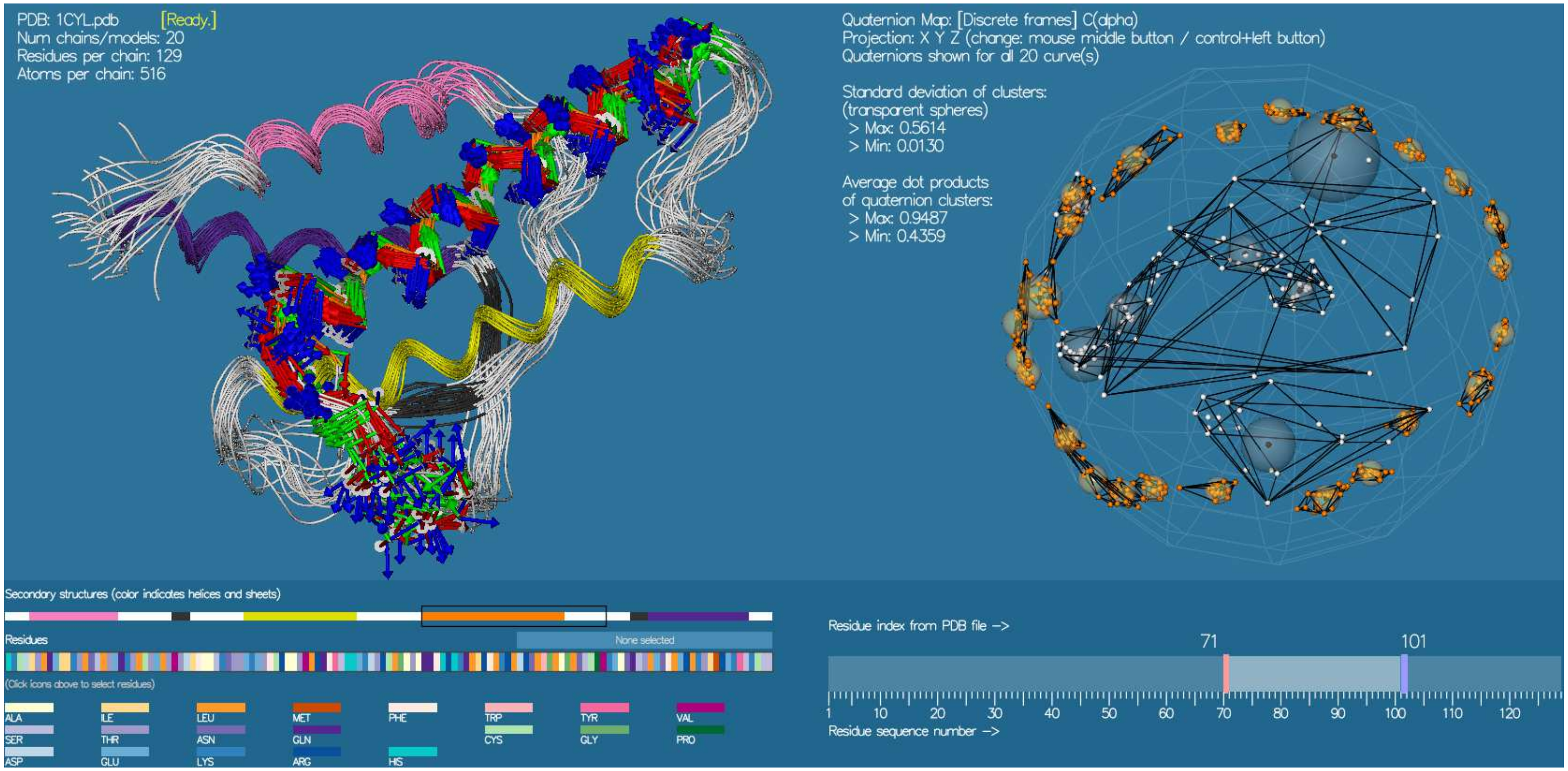
Dinner at Sushi Masuda (⋆⋆)
Tokyo, April 2017

In memory of **Tohru Eguchi:** February 2, 1948 – January 30, 2019

Other things I do — **My Book: Visualizing Quaternions**

Topics include visualizing the "Dirac String Trick," methods for interactive graphics of quaternions, applications to moving 3D coordinate frames, and many other applications and topics that could not have been imagined by William Rowan Hamilton, who discovered quaternions in 1843.

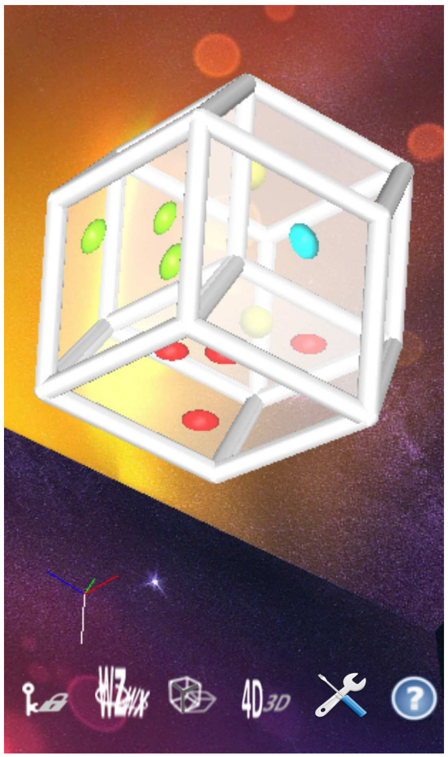
Quaternion Proteomics Applications



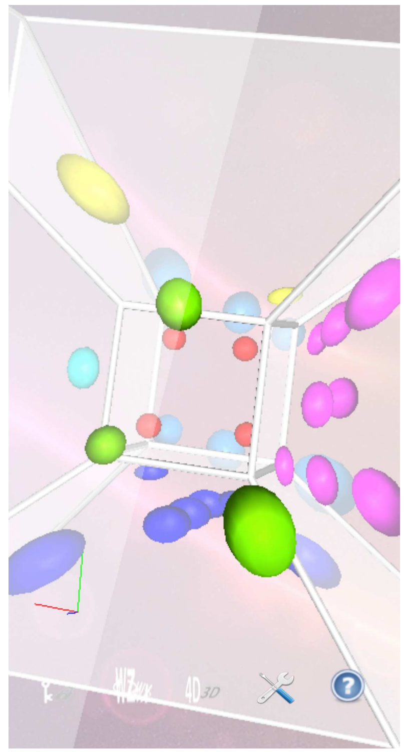
A.J. Hanson and S. Thakur, "Quaternion maps of global protein structure," Journal of Molecular Graphics and Modelling, Volume 38, September 2012, pp. 256–278.

My 4D Intuition-Friendly User Interfaces:

4Dice

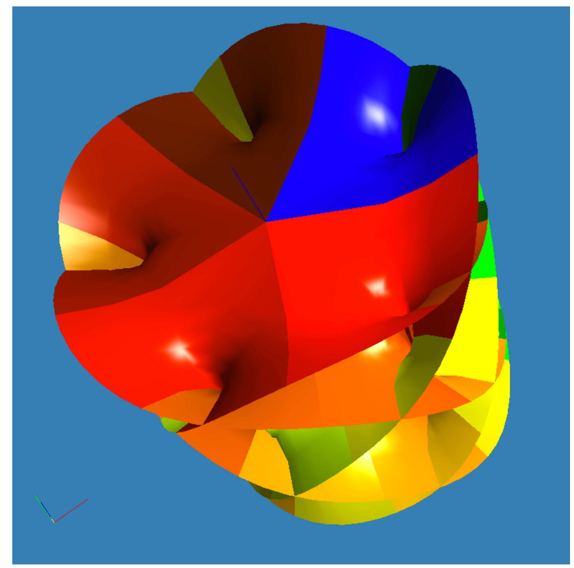


4DRoom



4D Explorer

4D Explorer



Free on the App Store! || `http://homes.sice.indiana.edu/hansona`

My Visualizations of Calabi-Yau Spaces

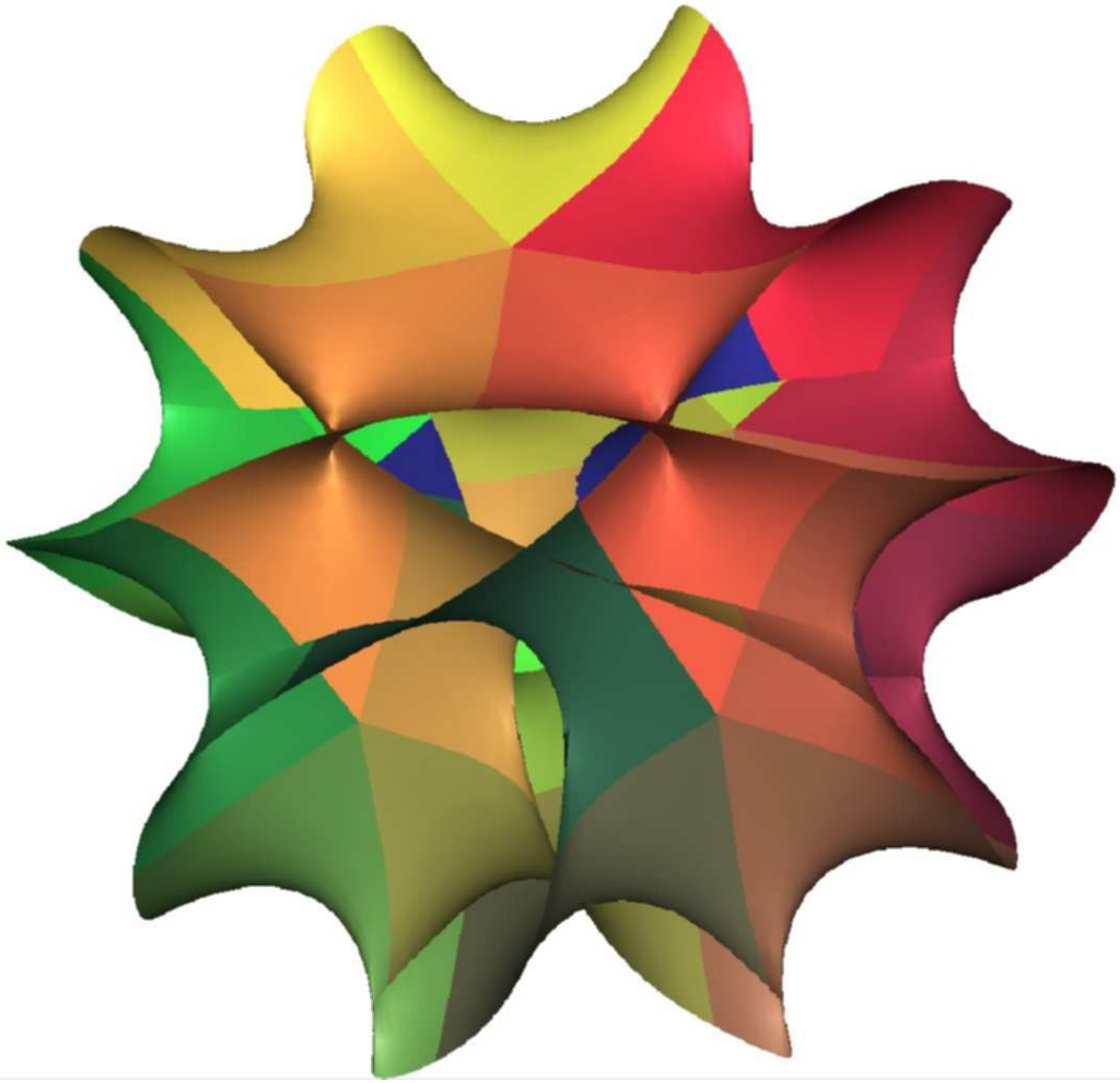
The **Calabi-Yau quintic** is a 6-manifold representing the hidden dimensions of 10D String Theory, and can be written as a quintic polynomial embedded in $\mathbb{C}\mathbf{P}(4)$:

$$z_0{}^5 + z_1{}^5 + z_2{}^5 + z_3{}^5 + z_4{}^5 = 0$$

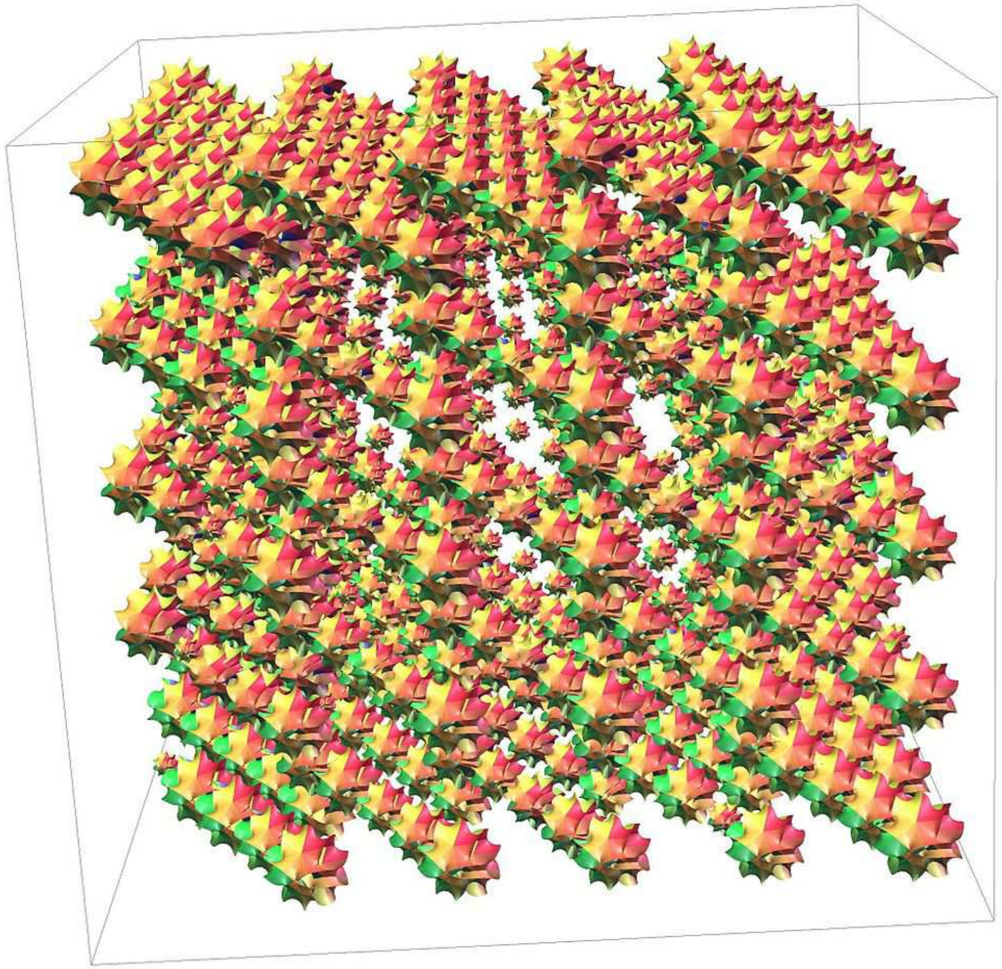
My 4D interfaces can be used to visualize the **2D cross-section** in $\mathbb{C}\mathbf{P}(2)$ of the Calabi-Yau quintic, a surface embedded in 4D, satisfying the equation

$$z_1{}^5 + z_2{}^5 = 1$$

Visualization of Calabi-Yau Quintic 2D Cross-Section

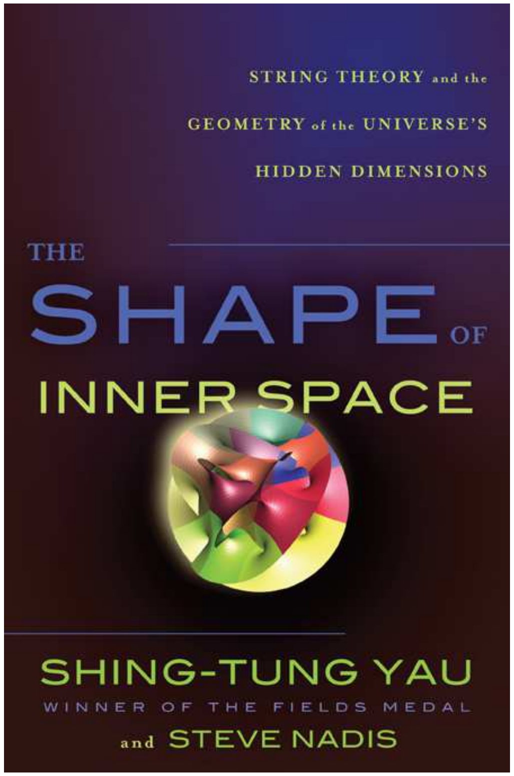


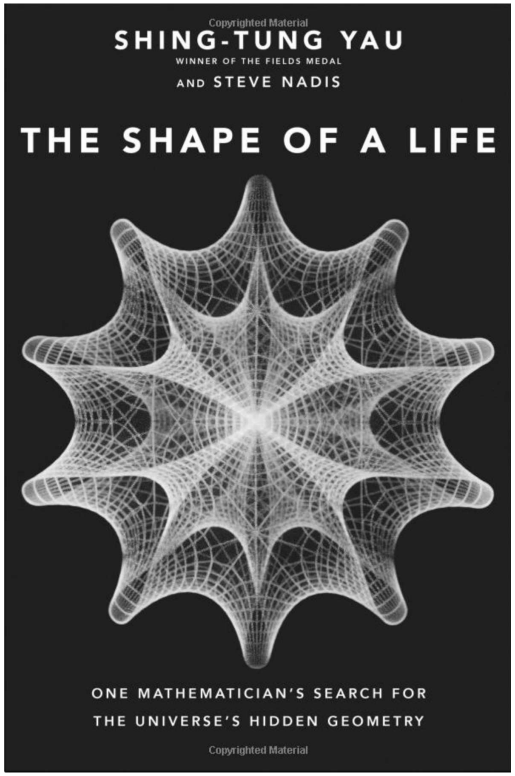
The Big Picture: The 6D Calabi-Yau Quintic Structure



This is actually SIX dimensional: the partial space is sampled on a 4D grid, and the remaining 2D cross-sections are shown as they change across the grid.

Cover art and logos for Shing-Tung Yau:





Visualizing the Eguchi-Hanson Metric

- History of how the metric came about
- General principles of metric visualization task
- The metric itself
- Approach to finding an isometric embedding
- Solving the embedding
- Exploit embedding to produce a visualization

The Eguchi-Hanson Metric: History

- **1975 – Yang-Mills Instanton:** The discovery of a simple asymptotically flat self-dual solution to the highly nonlinear $\mathrm{SU}(2)$ Yang-Mills equation was a surprise.
- **Early 1977 – Hawking proposes a candidate gravitational instanton:**. This looked promising due to its self-dual nature, but its asymptotic behavior was a possible issue.
- **1977 – Meeting Eguchi at SLAC:** I met Eguchi at SLAC in the fall of 1977, he shared some elegant techniques he had learned from Peter Freund while at the University of Chicago, and I became interested in the **Gravitational Instanton**, which Eguchi had already studied with Freund. Eguchi and I eventually recognized Hawking's proposed instanton as having the wrong asymptotic behavior (it was 3D flat), and started looking for a better solution with **4D flat** asymptotics.

History, contd.

- **Very early 1978 – almost a solution:** Using 4D polar coordinates and the Maurer-Cartan differential forms, we found a 4D-asymptotically-flat candidate, and wrote to Hawking that we thought we had a better way. But we did not publish because there was something wrong, a puzzling singularity.
- **January 1978 – meeting with Iz Singer:** I was at Berkeley at this time, and a very well-known mathematician, Isadore Singer, had recently come to Berkeley from MIT. We went to Singer, and he recognized our singularity as a standard problem with a standard solution in **under a minute** after we showed him the solution on the blackboard!
- **6 February 1978 – Gravity Instanton paper submitted to Phys Lett B:** We published our paper with the Eguchi-Hanson metric on April 10, 1978.

History, conclusion.

- **7 July 1978 – Gibbons and Hawking find the Multi-Instanton:** The story was completed half a year later when Gibbons and Hawking discovered a 4D-asymptotically-flat gravitational multi-instanton solution, very close to Hawking's 1977 3D-flat solution, labeled by an integer k, for which Eguchi-Hanson was the $k = 1$ case.

- **December 1980: Eguchi/Gilkey/Hanson:** In late 1979, Roman Jackiw invited us to write a review of the combination of ideas from Gravitation, Yang-Mills Gauge theories, and modern Differential Geometry and the relation to fiber bundles. This 181-page article was published in **Physics Reports** at the end of 1980, the last time that I published with Eguchi. The EGH paper has been cited well over 2000 times.

11D Nash embedding of self-dual Einstein metric

- For any given solution of Einstein's equations (in 4D Euclidean space for the ADE metrics, with Eguchi-Hanson being the A_1 metric), one wonders **can we visualize the actual geometry**?
- This question is often answered by searching for a **Nash embedding**, since Nash's theorem tells us that **an isometric embedding exists** for any Riemannian metric and its topological space, but Nash gave NO INFORMATION on how to FIND this embedding.
- So, with my collaborator, mathematician Ji-Ping Sha, we set out to investigate the possibility of a Nash embedding with a space whose **induced metric** was exactly the Eguchi-Hanson metric. **In the following, we discuss the solution we found in 11 Euclidean dimensions.**

The Eguchi-Hanson Metric Itself

- **Metric Ansatz:** We begin by assuming a 4D metric whose line element is of the form

$$(ds)^2 = f(r)^2 dr^2 + r^2({\sigma_x}^2 + {\sigma_y}^2) + r^2 g(r)^2 {\sigma_z}^2$$

  where the one-forms $\{\sigma_x, \sigma_y, \sigma_z\}$ form a basis for the Maurer-Cartan algebra, obeying

$$d\sigma_x + 2\sigma_y \wedge \sigma_z \;\; \textit{(cyclic)}.$$

  In the limit $f(r) = g(r) = 1$, this is simply the polar form of a flat Euclidean metric on $\mathbb{R}^4$.

- **Vierbein Decomposition:** Thus the metric can be written as

$$(ds)^2 = dx^\mu g_{\mu\nu} dx^\nu = e^a \delta_{ab} e^b,$$

  where the **vierbein one-forms** $e^a = {e^a}_\mu dx^\mu$ take the form

$$e^a = \{f(r)dr,\, r\sigma_x,\, r\sigma_y,\, rg(r)\sigma_z\}\ .$$

Eguchi-Hanson Metric, contd.

- **Connection 1-forms:** In the *vierbein* form, Riemannian geometry is very simple: we simply compute the torsion 2-form, set it to zero for the Levi-Civita connection condition, and solve for the connection 1-form ω_{ab}:

$$de^a + \omega^a{}_b \wedge e^b = 0$$

- **Self-dual Connection Condition:**. Then it can be shown that the Ricci tensor of the curvature 2-form

$$R_{ab} = d\omega_{ab} + \omega_{ac} \wedge \omega^c{}_b$$

will **vanish**, solving Einstein's equations, if ω_{ab} itself (which is a 4D Euclidean antisymmetric object, not requiring upper and lower index distinction) is **self-dual**:

$$\omega_{ab} = \epsilon_{abcd}\, \omega_{cd} \ ,$$

that is $\omega_{01} = \omega_{23}$, cyclic.

Eguchi-Hanson Metric, contd.

- **Eguchi-Hanson Solution:** The self-duality condition gives $fg = 1$ and a simple 1st-order differential equation for $f(r)$,

$$rf'(r) + 2f(r)\left(f(r)^2 - 1\right) = 0$$

whose solution is

$$f(r)^2 = \frac{1}{1 - \left(\frac{a}{r}\right)^4}.$$

The resulting connection 1-forms ω_{ab} are self-dual, the curvature 2-form R_{ab} is self-dual, and thus Einstein's equations are solved with an asymptotically flat metric.

- **Vierbein Solution for Metric:**

$$e^a = \left\{ \frac{1}{\sqrt{1 - \left(\frac{a}{r}\right)^4}} dr,\ r\sigma_x,\ r\sigma_y,\ \sqrt{1 - \left(\frac{a}{r}\right)^4}\, r\sigma_z \right\}$$

- **Removing Singularity at origin:** If the full angular range of S^3 is used for this metric at constant r, there is a cone singularity at $r = a$. This can be removed by *restricting* the volume measure to *one half* the range of S^3, giving asymptotic topology at infinity of **the Projective Space** $\mathbb{R}\mathbf{P}(3)$, which is isomorphic to the 3D rotations $\mathbf{SO}(3)$.

- **Topology is $\mathrm{T}^*\mathrm{S}^2$:** It can then be shown that the topology of the nonsingular Eguchi-Hanson Einstein manifold is the cotangent space of the 2D sphere S^2, which is S^2 at the "origin" $r = a$, and asymptotically $\mathbb{R}\mathbf{P}(3) = \mathbf{SO}(3)$ as $r \to \infty$.

- **Infinity is $\mathrm{SO}(3)$:** ***So now we know that any embedding must reduce to the topology of the 3D rotation group at infinity.***

Approach to Finding an Isometric Embedding

- **Write down $\mathrm{SO}(3)$:** Using the **quaternion** $q = (w, x, y, z)$ with $q \cdot q = r^2$, any element of the rotation group, topologically $\mathbb{R}\mathbf{P}(3)$, can be written

$$\frac{1}{r^2}\begin{bmatrix} w^2 + x^2 - y^2 - z^2 & 2xy - 2wz & 2xz + 2wy \\ 2xy + 2wz & w^2 - x^2 + y^2 - z^2 & 2yz - 2wx \\ 2xz - 2wy & 2yz + 2wx & w^2 - x^2 - y^2 + z^2 \end{bmatrix}$$

- **Columns are Hopf Fibrations $= \mathrm{S}^2$:** Every column and row is a 3D unit vector, with $\mathbf{v} \cdot \mathbf{v} = 1$, and thus a **two-sphere**. We exploit this to create the S^2 at the origin $r = a$ of $\mathbf{T}^*\mathbf{S}^2$.

Solving the Embedding Constraint Equations

- **Column Vector of $\mathbb{RP}(3)$:** Just map the elements of $q = (w,x,y,z)$, now with $q \cdot q = r^2$, to the *entire 3D rotation matrix*, first in $\mathbb{R}^{10}$, then extend to an $\mathbb{R}^{11}$ interpolation $\mathbf{p}(w,x,y,z)$ between $\mathbb{RP}(3)$ and the Hopf fibration to $\mathbf{S}^2$:

$\mathbb{RP}(3)$ Hopf $\mathbf{S}^2$ Interpolation

$$
\begin{bmatrix} w^2 \\ x^2 \\ y^2 \\ z^2 \\ wx \\ wy \\ yz \\ zx \\ wz \\ xy \end{bmatrix}
\to \frac{1}{r^2}
\begin{bmatrix} \frac{1}{\sqrt{2}}(w^2+z^2) \\ \frac{1}{\sqrt{2}}(x^2+y^2) \\ \frac{1}{\sqrt{2}}(y^2+x^2) \\ \frac{1}{\sqrt{2}}(z^2+w^2) \\ wx - yz \\ wy + xz \\ yz - wx \\ xz + wy \\ wz - wz = 0 \\ xy - xy = 0 \end{bmatrix}
\Rightarrow \mathbf{p} = \frac{1}{r^2}
\begin{bmatrix} \frac{1}{\sqrt{2}}(\alpha(r)w^2 + \beta(r)z^2) \\ \frac{1}{\sqrt{2}}(\alpha(r)x^2 + \beta(r)y^2) \\ \frac{1}{\sqrt{2}}(\alpha(r)y^2 + \beta(r)x^2) \\ \frac{1}{\sqrt{2}}(\alpha(r)z^2 + \beta(r)w^2) \\ \alpha(r)wx - \beta(r)yz \\ \alpha(r)wy + \beta(r)xz \\ \alpha(r)yz - \beta(r)wx \\ \alpha(r)xz + \beta(r)wy \\ (\alpha(r) - \beta(r))wz \\ (\alpha(r) - \beta(r))xy \\ \frac{1}{\sqrt{2}}\gamma(r) \end{bmatrix}
$$

Solving the Embedding Constraint Equations

- **Interpolation Limits: origin:** $\alpha(a) = \beta(a)$ is the S^2 topology as $r \to a$.

- **Interpolation Limits: infinity:** As $r \to \infty$, $\alpha(r) \to r$, and $\beta(r) \to 0$ gives the $\mathbb{R}\mathrm{P}(3)$ topology at ∞.

- **Rigidity in $\mathbb{R}^{10}$ requires an extra dimension:** We find that the $\mathbb{R}^{10}$ Ansatz, with $\gamma(r) = 0$ *fails*, as the geometry is too rigid for a successful Nash embedding. Adding **one more dimension**, to embed in $\mathbb{R}^{11}$ with a third interpolation function $\gamma(r)$ is sufficient for a successful Nash embedding!

The Hopf fiber $\mathbf{S}^2$ limit in $\mathbb{R}^{11}$

- **Column Vector of $\mathbb{R}\mathbf{P}(3)$:** At $q \cdot q = a^2$, where $\alpha(a) = \beta(a)$, we have a map from $\mathbb{R}\mathbf{P}(3)$ to the $\mathbf{S}^2$ Hopf fibration:

$$\mathbf{p}(r = a) = \frac{1}{r^2} \begin{bmatrix} \frac{1}{\sqrt{2}}(w^2 + z^2) \\ \frac{1}{\sqrt{2}}(x^2 + y^2) \\ \frac{1}{\sqrt{2}}(y^2 + x^2) \\ \frac{1}{\sqrt{2}}(z^2 + w^2) \\ wx - yz \\ yz - wx \\ wy + xz \\ xz + wy \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}}\gamma(a) \end{bmatrix}$$

- **A degenerate pair of spheres:** The result at $r = a$ is just a single sphere echoed in the $(1, 3, 5, 7)$ and the $(2, 4, 6, 8)$ coordinates of $\mathbb{R}^{11}$.

Identify Induced Metric with the Known Metric:

- **Induced metric:**. Using the 4D $q = (w, x, y, z)$, $q \cdot q = r^2$, coordinates to compute the induced metric from the interpolation,

$$g_{\mu\nu} = \sum_{i=1}^{11} \frac{\partial \mathbf{p}_i(w,x,y,z)}{\partial q^\mu} \frac{\partial \mathbf{p}_i(w,x,y,z)}{\partial q^\nu}$$

and extracting the correspondence with $(\alpha(r), \beta(r), \gamma(r))$, we find

$$\alpha(r) = \frac{\sqrt{r^4+\sqrt{r^8-a^8}}}{\sqrt{2}r} \qquad \beta(r) = \frac{a^4}{\sqrt{2}r\sqrt{r^4+\sqrt{r^8-a^8}}}$$

$$\gamma'(r) = \sqrt{\frac{3a^4 + r^4}{a^4 + r^4}}$$

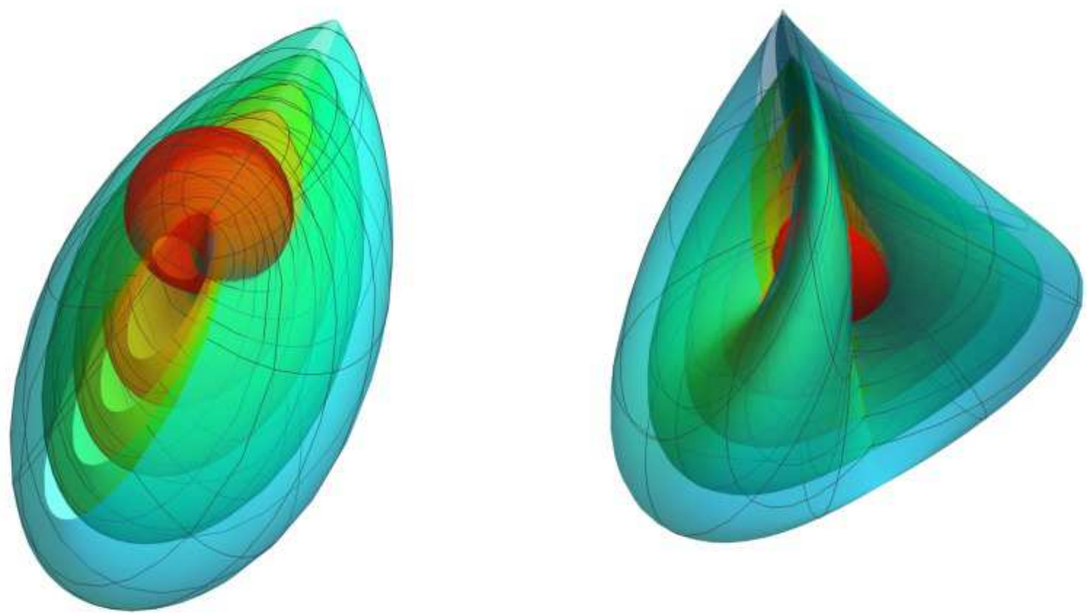
Visualization of the Eguchi-Hanson Space

- **The Nash Embedding $\mathbf{p}(w, x, y, z)$:** The 4-manifold in $\mathbb{R}^{11}$ parameterized by the vector $q = (w, x, y, z)$ with $|q| = r$ is analogous to a round sphere with a metric of constant curvature: the induced metric with the solutions for $(\alpha(r), \beta(r), \gamma'(r))$ given above is the **shape of the space** on which the metric lives, our Nash embedding for the Eguchi-Hanson Space.

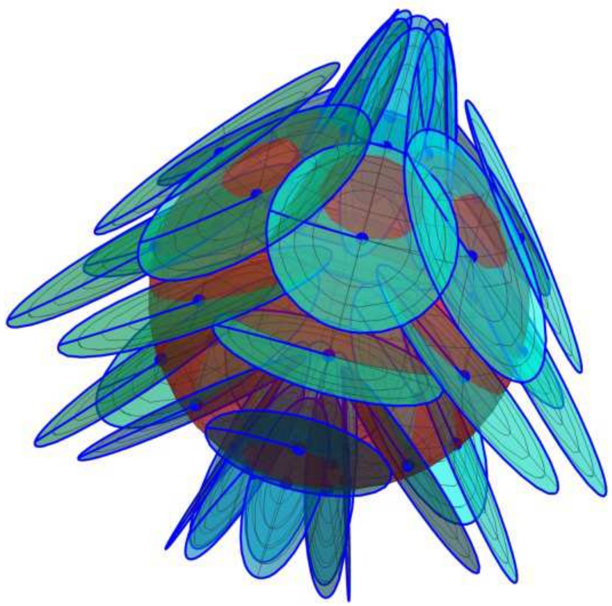
- **Pick 2D $\times$ 2D samplings from $\mathbf{p}(w, x, y, z)$ in $\mathbb{R}^{11}$ to subspaces in $\mathbb{R}^3$:** We can make computer graphics images of the embedding $\mathbf{p}(w, x, y, z)$ by first choosing a 2D family of sample points, e.g., on the $\mathbf{S}^2$ at the origin $r = a$, then plotting the *remaining* 2D subsurface coordinates at *each sample point*.

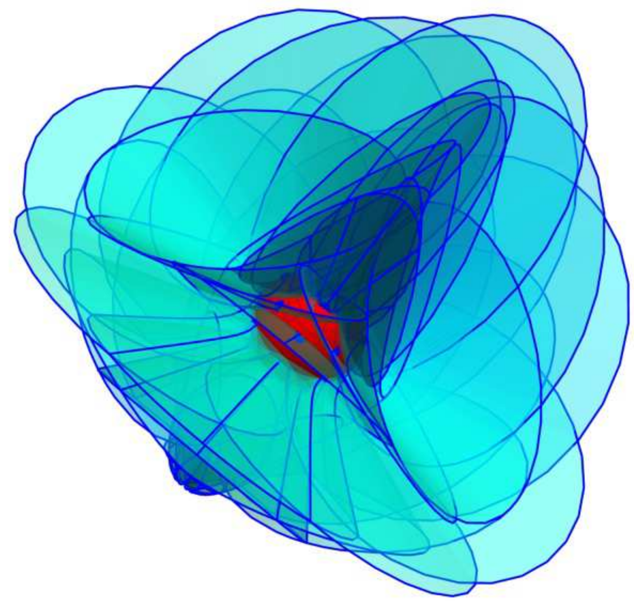
- **Pick projections from $\mathbf{p}(w, x, y, z)$ subspaces to plot in $\mathbb{R}^3$:** Now we have surfaces to plot, and we choose 11×3 matrices to project each 11D point to computer-graphics compatible 3D point for rendering. **These results, for carefully chosen projection matrices, are given next.**

Visualizations of T^*S^2 Projected 11D $\to$ 3D



... more Visualizations of $\mathbf{T}^*\mathbf{S}^2$ Projected 11D $\rightarrow$ 3D





Conclusion of our Journey:

- **Introduction: my work, from Einstein metrics to Quaternions to 4D visualization to Calabi-Yau space images.**

- **History of the Eguchi-Hanson Metric.**

- **Solving the Isometric Embedding Problem for Eguchi-Hanson Space.**

- **Exploiting the Embedding to Produce Exact Computer Graphics Images of the 4D Manifold Whose Induced Metric is the Eguchi-Hanson A_1 Einstein Metric.**

Thank you!

For more information on my work, see my web page

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