Isometric Embedding of the A_1 Gravitational Instanton

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We explore embeddings that lead to Ricci-flat metrics on $\mathbf{T}^*\mathbf{S}^2$ corresponding to the A_1 (Eguchi-Hanson) self-dual, asymptotically locally Euclidean, gravitational instanton. The variety of ways such embeddings appear reveals a spectrum of intuitions about the geometric structure of such spaces. Our main result is a Ricci-flat isometric embedding of $\mathbf{T}^*\mathbf{S}^2$ in Euclidean $\mathbb{R}^{11}$, starting with an embedded $\mathbb{R}\mathbf{P}^3$ manifold and then interpolating in a radial variable to an $\mathbf{S}^2$ at the origin via a Hopf fibration. We discover an interpolation yielding an embedded 4-manifold whose induced metric is precisely the Eguchi-Hanson form. This leads us to investigate other embeddings known from the theory of hyperkähler moment maps that have deterministic relations to Ricci-flat metrics for A_1, but have less obvious identifications with $\mathbf{T}^*\mathbf{S}^2$. A canonical approach suggested by Hitchin starts with a set of three (real) moment-map constraints on $\mathbb{C}^4$ as the basis for an A_1 hyperkähler quotient. We exhibit an explicit vector in $\mathbb{C}^4$ with 4 real parameters that satisfies the three moment-map constraints but does not induce a Ricci-flat metric. However, if we add a constraint-invariant gauge as a 5th real parameter, the 5-dimensional metric pulled back from the gauged embedding reduces to the desired 4D Ricci-flat metric on $\mathbf{T}^*\mathbf{S}^2$ when we quotient by the $\mathbf{U}(1)$ of the gauge parameter. Finally, we review the results of Alvarez-Gaumé and Freedman, who use yet another embedding of the moment-map constraints in holomorphic variables in $\mathbb{C}^4$ to produce a Kähler potential for a Ricci-flat A_1 metric in the context of $N = 2$ supersymmetric σ models. Their method produces a Kähler potential for a Ricci-flat metric without the using an explicit quotient step. The constructions of solutions to slightly different embedding problems that we have presented here all generate Ricci-flat metrics for the A_1 instanton on $\mathbf{T}^*\mathbf{S}^2$, but depend in distinct ways on the underlying geometric properties of the spaces employed.

1. Introduction

Our goal here is to explore embeddings of manifolds in Euclidean spaces related to the metrics of the simplest so-called "gravitational instanton," which is a Ricci-flat, asymptotically locally Euclidean (ALE), Einstein metric on the topological 4-manifold $\mathbf{T}^*\mathbf{S}^2$. This metric corresponds to the $k = 1$ case of the A_k (cyclic group) gravitational instanton series, and has a self-dual Riemann tensor (we will

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not distinguish here between self-dual and anti-self-dual). The first example is the Eguchi-Hanson[1] (1978) metric, which is also known to be the $n = 1$ case of a family of $2n$-complex-dimensional Ricci-flat spaces described by Calabi[2] (1979). Subsequently Gibbons and Hawking[3] (1978) produced a series of ALE Einstein metrics, parameterized by $k + 1$ 3D points, that turned out to describe each of the possible A_k instantons, with the $k = 1$ case corresponding precisely to the Eguchi-Hanson metric (see, e.g., Prasad[4]).

The first realization that there could be a correspondence between *all* the discrete groups of $\mathbf{SU}(2)$ acting on $\mathbf{S}^2$, characterized by the Kleinian (ADE) polynomials,[5] and the class of asymptotically locally Euclidean (ALE) self-dual Einstein spaces was due to Hitchin, in his landmark "Polygons and Gravitons" paper[6] (1979), appearing immediately after the work of Eguchi-Hanson and Gibbons-Hawking. Hitchin conjectured but could not prove that the association of Einstein manifolds to the two infinite series of discrete $\mathbf{SU}(2)$ subgroups, A_k (cyclic) and D_k (dihedral), along with the three discrete examples, E_6 (tetrahedral), E_7 (octahedral), and E_8 (icosahedral), exhausted the possible ALE gravitational instantons. After some years of continuing interest based on the relationship discovered between the ADE metrics and supersymmetry, Lindström and Roček[7] introduced a basic version of the hyperkähler quotient method in 1983. As further insights developed that revealed associations among supersymmetry, hyperkähler quotients, Dynkin diagrams, quiver diagrams, twistor methods, and the ADE spaces, Hitchin, Karlhede, Lindström, and Roček[8] ultimately were able in 1986 to draw together the various pieces into a coherent mathematical framework incorporating hyperkähler quotient methods as well as twistor methods. Finally, in 1989, Kronheimer[9] was able to wrap things up and show, via twistor methods and the hyperkähler quotient mechanism, that the ADE spaces indeed exhausted the possible ALE self-dual Einstein spaces. Among subsequent investigations, Lindström, Roček, and von Unge[10] showed how, starting directly with quiver diagrams and the implied moment-map constraints, one could explicitly derive not just the Kleinian ADE algebraic varieties, but also the more complicated singularity-resolved forms associated with du Val.[11–13]

Within this elaborate context, we will explore embedded submanifolds in Euclidean $\mathbb{R}^N$ and $\mathbb{C}^N$ that are related to the A_1 gravitational instanton either via an induced metric, or through hyperkähler moment map constraints. One may note that the underlying manifold of the A_1 gravitational instanton is diffeomorphic to $\mathbf{T}^*\mathbf{S}^2$ (the cotangent bundle of the 2-sphere $\mathbf{S}^2$), which is naturally embedded in $\mathbb{R}^6$. However, in its standard form, the induced metric of this embedding is not Ricci-flat. We resolve this by constructing an explicit smooth parametric embedding of $\mathbf{T}^*\mathbf{S}^2$ in $\mathbb{R}^{11}$. Then it is straightforward, by appropriately choosing the interpolation functions in this embedding, to find an induced metric that corresponds exactly to the Ricci-flat ALE Eguchi-Hanson metric.

Our treatment is organized as follows: After some introductory context material,

we construct our new parametric embedding of $\mathbf{T}^*\mathbf{S}^2$ in $\mathbb{R}^{11}$. The resulting induced metric corresponds exactly to the form found by Eguchi-Hanson,[1] and exposes the interpolation functions to be identified with the Ricci-flat solution, explicitly solving the isometric embedding problem. This enables us to produce novel pictures of the geometric features of this embedding. Next, we study the $\mathbb{C}^4$ ($\mathbb{R}^8$) space of the A_1 hyperkähler quotient framework suggested by Hitchin,[14] and present a real 5-parameter solution of the three moment-map constraints, along with an explicit quotient by $\mathbf{U}(1)$ down to a 4D Einstein metric, yielding an equivalent but distinct Ricci-flat metric on $\mathbf{T}^*\mathbf{S}^2$. We conclude by reviewing an approach to the A_1 solution due to Alvarez-Gaumé and Freedman,[15] which uses an alternate 2-complex-parameter holomorphic embedding of a different moment-map constraint condition in $\mathbb{C}^4$ ($\mathbb{R}^8$) and leads to an elegant Kähler potential for the Ricci-flat A_1 metric.

2. Features of the Eguchi-Hanson A_1 metric

We begin with a review of some of the structures we will refer to in the process of building an isometric embedding for the Ricci-flat A_1 metric. We will be looking for collections of coefficients of the Maurer-Cartan forms. We will write these in terms of the generic quaternion element g parameterized by the point $\{w, x, y, z\}/r$ in $\mathbf{SU}(2)$ (or $\mathbf{S}^3$) as

$$g = \frac{1}{r}\left(wI_2 - \mathrm{i}\,\mathbf{\Sigma}\cdot[x,\,y,\,z]\right)$$
$$= \frac{1}{r}\begin{bmatrix} w - \mathrm{i}\,z & -\mathrm{i}\,x - y \\ y - \mathrm{i}\,x & w + \mathrm{i}\,z \end{bmatrix},$$

where $\mathbf{\Sigma}$ denotes the conventional Pauli matrices (we reserve the symbol σ for other purposes below), and $r^2 = w^2 + x^2 + y^2 + z^2$. The form of g is chosen to reproduce precisely a standard right-handed quaternion algebra when two distinct matrices are multiplied together. We then define the Maurer-Cartan forms in our Euclidean coordinates as

$$g^{-1}dg = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix} = \frac{1}{r^2}\begin{bmatrix} -x & w & z & -y \\ -y & -z & w & x \\ -z & y & -x & w \end{bmatrix}\cdot\begin{bmatrix} dw \\ dx \\ dy \\ dz \end{bmatrix}.$$

We note for reference the commonly used polar form (see Eq. (7)) for the Maurer-Cartan forms:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(\cos\psi\,d\theta + \sin\psi\,\sin\theta\,d\phi\right) \\ \frac{1}{2}\left(-\sin\psi\,d\theta + \cos\psi\,\sin\theta\,d\phi\right) \\ \frac{1}{2}\left(d\psi + \cos\theta\,d\phi\right) \end{bmatrix}.$$

These forms obey the fundamental structure equations

$$d\sigma_x + 2\sigma_y \wedge \sigma_z = 0$$
$$d\sigma_y + 2\sigma_z \wedge \sigma_x = 0$$
$$d\sigma_z + 2\sigma_x \wedge \sigma_y = 0\ .$$

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(This follows from taking the exterior derivative of $I = g^{-1}g$ and observing that $d(g^{-1}dg) + g^{-1}dg \wedge g^{-1}dg = 0$; the factor of 2, an algebraic identity in the 1-forms, comes from the sum over Pauli matrices in the matrix form of g.) The attentive reader will note that there are some alternate choices of signs and factors of 2 in the literature;[16,17] this is our chosen convention for this presentation.

2.1. *Writing out the solution in an* **SU(2)** *basis*

If one takes a Maurer-Cartan basis of 1-forms for $\mathbf{SU}(2)$ using the above form for the σ's, then the Eguchi-Hanson solution for the self-dual metric can be written as $d\tau^2 = (e_0)^2 + (e_1)^2 + (e_2)^2 + (e_3)^2$ with the following vierbeins,

$$e = \left\{ \frac{1}{\sqrt{1 - \left(\frac{s}{r}\right)^4}} dr, \ r\sigma_x, \ r\sigma_y, \ \sqrt{1 - \left(\frac{s}{r}\right)^4} \, r\sigma_z \right\} , \tag{1}$$

and s a constant. The connection 1-forms $\omega^a{}_b$ are the solutions to the Levi-Civita torsion-free conditions $de^a + \omega^a{}_b \wedge e^b = 0$. In this gauge, the solutions for the connections themselves take the self-dual form

$$\omega_x = \omega_{23} = \omega_{01} = -\sqrt{1 - \left(\frac{s}{r}\right)^4} \, \sigma_x$$

$$\omega_y = \omega_{31} = \omega_{02} = -\sqrt{1 - \left(\frac{s}{r}\right)^4} \, \sigma_y \tag{2}$$

$$\omega_z = \omega_{12} = \omega_{03} = -\left(1 + \left(\frac{s}{r}\right)^4\right) \sigma_z ,$$

while the Riemann curvature 2-forms $R^a{}_b = d\omega^a{}_b + \omega^a{}_c \omega^c{}_b$ take the form

$$R_x = R_{23} = R_{01} = -\frac{2s^4}{r^6} \left(e^2 \wedge e^3 + e^0 \wedge e^1\right)$$

$$R_y = R_{31} = R_{02} = -\frac{2s^4}{r^6} \left(e^3 \wedge e^1 + e^0 \wedge e^2\right) \tag{3}$$

$$R_z = R_{12} = R_{03} = +\frac{4s^4}{r^6} \left(e^1 \wedge e^2 + e^0 \wedge e^3\right) .$$

The explicit self-duality of the connection 1-forms $\omega^a{}_b$ produces an automatically self-dual Riemann curvature 2-form, and that in turn implies the vanishing of the Ricci tensor, so this is a Ricci-flat Einstein space in four Euclidean dimensions. Observe that changing the order of $\{w, x, y, z\}$ or the sign of $\{\sigma_x, \sigma_y, \sigma_z\}$ can change various signs in Eq. (2) and Eq. (3) and can interchange self-dual and anti-self-dual labeling. Restricting the parameters to $\mathbb{R}\mathbf{P}^3$ instead of a Euclidean $\mathbf{S}^3$ at infinity removes the cone singularity at the core $\mathbf{S}^2$ as $r \to s$, and one can verify by explicit integration that, with that choice of integration volume, the Euler integral of the volume is $3/2$, while the Chern surface term is $1/2$, giving the total Euler number $\chi = 2$ as required by the topology of $\mathbf{T}^*\mathbf{S}^2$; the signature, which has no surface correction, can similarly be shown to be $\tau = -1$.

2.2. *The Eguchi-Hanson template form*

For the purposes of interpreting our embedding in the next section, we now rewrite the Eguchi-Hanson solution Eq. (1) as an abstraction (which in fact is a general Bianchi type IX metric) of the form

$$e = \left\{ \sqrt{f(r)}dr,\, r\sqrt{g(r)}\sigma_x,\, r\sqrt{g(r)}\sigma_y,\, r\sqrt{h(r)}\sigma_z \right\}\,, \tag{4}$$

where

$$f(r) = \left(1 - \left(\tfrac{s}{r} \right)^4 \right)^{-1} \qquad g(r) = 1 \qquad h(r) = \left(1 - \left(\tfrac{s}{r} \right)^4 \right)\,. \tag{5}$$

From $d\tau^2 = e \cdot e = dx^\mu g_{\mu\nu} dx^\nu$, we can extract a convenient Cartesian form of the metric that we will be able to match with the Cartesian form of our anticipated embedding, with coordinates $x^\mu = \{w, x, y, z\}$ and $r^2 = w^2 + x^2 + y^2 + z^2$:

$g_{\mu\nu} =$

$$\frac{1}{r^2}\left[\begin{array}{cc} w^2 f(r) + \left(x^2+y^2\right)g(r) + z^2 h(r) & wxf(r) + (-wx+yz)\,g(r) - yzh(r) \\ wxf(r) + (-wx+yz)\,g(r) - yzh(r) & x^2 f(r) + \left(w^2+z^2\right)g(r) + y^2 h(r) \\ wyf(r) - (wy+xz)\,g(r) + xzh(r) & xy(f(r)-h(r)) \\ wz(f(r)-h(r)) & xzf(r) - (wy+xz)\,g(r) + wyh(r) \end{array} \right.$$

$$\left. \begin{array}{cc} wyf(r) - (wy+xz)\,g(r) + xzh(r) & wz(f(r)-h(r)) \\ xy(f(r)-h(r)) & xzf(r) - (wy+xz)\,g(r) + wyh(r) \\ y^2 f(r) + \left(w^2+z^2\right)g(r) + x^2 h(r) & yzf(r) + (wx-yz)\,g(r) - wxh(r) \\ yzf(r) + (wx-yz)\,g(r) - wxh(r) & z^2 f(r) + \left(x^2+y^2\right)g(r) + w^2 h(r) \end{array} \right]\,. \tag{6}$$

Using the polar coordinates

$$w = r\cos\frac{\theta}{2}\cos\frac{\phi+\psi}{2}$$

$$x = r\sin\frac{\theta}{2}\cos\frac{\phi-\psi}{2}$$

$$y = r\sin\frac{\theta}{2}\sin\frac{\phi-\psi}{2} \tag{7}$$

$$z = r\cos\frac{\theta}{2}\sin\frac{\phi+\psi}{2}$$

with the order $\{r, \theta, \phi, \psi\}$, $0 \le \theta < \pi$, $0 \le \phi < 2\pi$, $0 \le \psi < 2\pi$ (for $\mathbf{S}^3$, we would have $0 \le \psi < 4\pi$), we can write the metric in polar form as

$$\left[\begin{array}{cccc} f(r) & 0 & 0 & 0 \\ 0 & \tfrac{1}{4}r^2 g(r) & 0 & 0 \\ 0 & 0 & \tfrac{1}{4}r^2\left(h(r)\cos^2\theta + g(r)\sin^2\theta\right) & \tfrac{1}{4}r^2 h(r)\cos\theta \\ 0 & 0 & \tfrac{1}{4}r^2 h(r)\cos\theta & \tfrac{1}{4}r^2 h(r) \end{array} \right]\,. \tag{8}$$

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3. Embedding $\mathbf{T}^*\mathbf{S}^2$ (the A_1 Manifold) in $\mathbb{R}^{11}$

We now are ready to begin the process of deriving the metric of the A_1 member of the ADE family of Ricci-flat Euclidean Einstein spaces via an embedding in 11-dimensional Euclidean space. Our result corresponds exactly to the Eguchi-Hanson[1] form for the A_1 metric, which has properties quite different from the $k = 1$ form of the Gibbons-Hawking[3,4] $(k + 1)$-point parameterized solution for the whole family that has subsequently become known as the A_k gravitational instanton metrics. Starting out with *only* the topological space $\mathbf{T}^*\mathbf{S}^2$, using no other knowledge, we now show how to derive an isometric embedding of $\mathbf{T}^*\mathbf{S}^2$ in the Euclidean space $\mathbb{R}^{11}$ that corresponds to the Ricci-flat metric.

3.1. *Construction of the map*

We first need to find an embedding of $\mathbb{R}\mathbf{P}^3$, or equivalently $\mathbf{SO}(3)$, which is the asymptotic boundary of $\mathbf{T}^*\mathbf{S}^2$, along with some way of expressing the fact that $\mathbf{T}^*\mathbf{S}^2$ collapses topologically to $\mathbf{S}^2$ at the origin. One example of an embedding of $\mathbf{SO}(3)$ is the standard quadratic form mapping the unit quaternions parameterizing $\mathbf{S}^3$ to an $\mathbf{SO}(3)$ rotation matrix. A standard form for this matrix is a rotation by an angle θ about a fixed unit-norm axis $\hat{\mathbf{n}} = [n_1, n_2, n_3]$,

$$\mathbf{R}(\theta, \hat{\mathbf{n}}) = \begin{bmatrix} c + (n_1)^2(1-c) & n_1 n_2(1-c) - s n_3 & n_1 n_3(1-c) + s n_2 \\ n_2 n_1(1-c) + s n_3 & c + (n_2)^2(1-c) & n_2 n_3(1-c) - s n_1 \\ n_3 n_1(1-c) - s n_2 & n_3 n_2(1-c) + s n_1 & c + (n_3)^2(1-c) \end{bmatrix} , \quad (9)$$

where $c = \cos\theta$, $s = \sin\theta$, and $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1$ by definition. This is exactly equivalent to a quadratic quaternion form constructed from the unit quaternion $q = \{w, x, y, z\}$ obeying $q \cdot q = 1$ $(q \in \mathbf{S}^3)$, which takes the form

$$\mathbf{R}(q) = \begin{bmatrix} w^2 + x^2 - y^2 - z^2 & 2xy - 2wz & 2xz + 2wy \\ 2xy + 2wz & w^2 - x^2 + y^2 - z^2 & 2yz - 2wx \\ 2xz - 2wy & 2yz + 2wx & w^2 - x^2 - y^2 + z^2 \end{bmatrix} , \quad (10)$$

as one can confirm by substituting

$$q = \{\cos(\theta/2), \, n_1 \sin(\theta/2), \, n_2 \sin(\theta/2), \, n_3 \sin(\theta/2)\} . \quad (11)$$

We could of course use any convenient parameterization of $\mathbf{S}^3$ in place of Eq. (11); we will start with the Cartesian coordinates $q = \{w, x, y, z\}$ for notational convenience.

Since $\mathbf{R}(q)$ is an orthogonal matrix, each row and each column has unit length when $q \cdot q = 1$ is imposed, and all pairs of rows as well as pairs of columns are orthogonal. The important fact is that, since topologically $\mathbf{R}(q)$ is just $\mathbb{R}\mathbf{P}^3$, the nine quadratic expressions that make up $\mathbf{R}(q)$ are precisely a parameterized embedding mapping $\mathbf{S}^3$ to $\mathbb{R}\mathbf{P}^3$ in $\mathbb{R}^9$, or, more generally, in $\mathbb{R}^{10}$ if we relax $q \cdot q = 1 \to r^2$ to allow scaling. (Note that $\mathbf{S}^3$ double covers $\mathbb{R}\mathbf{P}^3$, so we must ultimately restrict the parameter domain of $\mathbf{S}^3$ accordingly.)

Next we untangle the combinations of quadratic forms in $\mathbf{R}(q)$ in the style of a Veronese map, and use the 10 individual quadratic forms to define an initial map from $\mathbf{S}^3$ to $\mathbb{R}\mathbf{P}^3$ embedded in $\mathbb{R}^{10}$ as follows:

$$
\mathbf{p}(w,x,y,z) = \begin{bmatrix} w^2 \\ x^2 \\ y^2 \\ z^2 \\ wx \\ wy \\ wz \\ yz \\ zx \\ xy \end{bmatrix} \qquad \hat{\mathbf{p}}(w,x,y,z) = \begin{bmatrix} w^2 \\ x^2 \\ y^2 \\ z^2 \\ \sqrt{2}wx \\ \sqrt{2}wy \\ \sqrt{2}wz \\ \sqrt{2}yz \\ \sqrt{2}zx \\ \sqrt{2}xy \end{bmatrix} , \qquad (12)
$$

The scaled version $\hat{\mathbf{p}}$ is constructed to define a radius of constant length, $\hat{\mathbf{p}} \cdot \hat{\mathbf{p}} = \left(w^2 + x^2 + y^2 + z^2\right)^2 = r^4$, so it is explicitly a point on a round $\mathbf{S}^9$.

Now that we have $\mathbb{R}\mathbf{P}^3$ embedded to define the asymptotic boundary of $\mathbf{T}^*\mathbf{S}^2$, as well as the ability to scale it into the interior, we must find a way to terminate that ingoing mapping smoothly on the "origin," which is topologically $\mathbf{S}^2$. That should be easy, considering the fact that we know another very nice map, the Hopf fibration, from $\mathbf{S}^3$ to either $\mathbb{R}^3$ or $\mathbb{R}^4$, whose image is an embedded $\mathbf{S}^2$. In fact we have six explicit such maps embodied in Eq. (10), since each column and each row has unit length and hence is a map from the three Euler angles of $\mathbf{S}^3$ to a two-parameter $\mathbf{S}^2$. We choose our map from the elements of the last column of Eq. (10),

$$
\text{(last column)} \quad \mathbf{m}(w,x,y,z) = \begin{bmatrix} \sqrt{2}(w^2 + z^2) \\ \sqrt{2}(x^2 + y^2) \\ 2(wx - yz) \\ 2(wy + xz) \end{bmatrix} , \qquad (13)
$$

where $(1/2)\mathbf{m} \cdot \mathbf{m} = \left(w^2 + x^2 + y^2 + z^2\right)^2 = r^4$, defining an $\mathbf{S}^2$ of radius $R = q \cdot q = r^2$. For reasons that will become clear, we have used the 4D version of the Hopf fibration that corresponds to the last column of Eq. (10) after a rotation by $45°$ to the desired axis component $w^2 + z^2 - (x^2 + y^2)$ of the 3D column's subspace (the remaining, orthogonal, direction is just $w^2 + x^2 + y^2 + z^2$). We check that this is indeed the fibration corresponding to fibering out the ψ angular variable in Eq. (7) by explicit substitution:

$$
\mathbf{R}_4(\frac{\pi}{4}, \text{1-2 plane}) \cdot \mathbf{m}(r,\theta,\phi,\psi) = r^2 \begin{bmatrix} \cos\theta \\ 1 \\ \cos\phi \sin\theta \\ \sin\phi \sin\theta \end{bmatrix} . \qquad (14)
$$

These are the standard $\mathbf{S}^2$ spherical coordinates $\{\cos\theta, \cos\phi \sin\theta, \sin\phi \sin\theta\}$, which we need at the "origin" of $\mathbf{T}^*\mathbf{S}^2$ in the embedding.

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We now proceed to generate a parameterized interpolation from Eq. (12) to Eq. (13) that will become the sought-for isometric embedding of $\mathbf{T}^*\mathbf{S}^2$. However, it turns out that 10 dimensions is actually just slightly too rigid to get an isometric embedding of the metric in this context, and we will have to add an eleventh dimension that we take to be parameterized by the scaling radius r. We thus consider this map, interpolating from $\mathbb{R}\mathbf{P}^3$ to $\mathbf{S}^2$ at the "origin" $r = s$, which we will use to pull back a metric on the 4-dimensional manifold parameterized by $\{w, x, y, z\}$:

$$
\mathbf{p}(w, x, y, z) = \frac{1}{r^2}
\begin{bmatrix}
\frac{1}{\sqrt{2}}(a(r)w^2 + b(r)z^2) \\
\frac{1}{\sqrt{2}}(a(r)x^2 + b(r)y^2) \\
\frac{1}{\sqrt{2}}(a(r)y^2 + b(r)x^2) \\
\frac{1}{\sqrt{2}}(a(r)z^2 + b(r)w^2) \\
a(r)wx - b(r)yz \\
a(r)yz - b(r)wx \\
a(r)wy + b(r)xz \\
a(r)xz + b(r)wy \\
(a(r) - b(r))wz \\
(a(r) - b(r))xy \\
\frac{1}{\sqrt{2}}c(r)
\end{bmatrix} .
\tag{15}
$$

If $a(r)$ and $b(r)$ are two monotonic positive smooth functions defined for $s \leq r < \infty$ with the properties that $a(s) = b(s)$, $a(r \to \infty) \to r$, and $b(r \to \infty) \to 0$, then we should have a smooth embedding of $\mathbf{T}^*\mathbf{S}^2$ incorporating interpolation functions in r that could permit deformation of the path in a way that produces a Ricci-flat induced metric.

As $r \to \infty$ (with $a(r) \to r$ and $b(r) \to 0$), the entire column corresponds to the $\mathbb{R}\mathbf{P}^3$ boundary. In the $\mathbf{S}^2$ limit, $a(s) = b(s)$, we can check using our polar coordinates from Eq. (7) that we have a Hopf fibered $\mathbf{S}^2$ embedded in the 11 vector components,

$$
\left\{ \frac{1}{\sqrt{2}}\cos^2\frac{\theta}{2}, \frac{1}{\sqrt{2}}\sin^2\frac{\theta}{2}, \frac{1}{\sqrt{2}}\sin^2\frac{\theta}{2}, \frac{1}{\sqrt{2}}\cos^2\frac{\theta}{2}, \right.
$$
$$
\left. \frac{1}{2}\cos\phi\sin\theta, -\frac{1}{2}\cos\phi\sin\theta, \frac{1}{2}\sin\phi\sin\theta, \frac{1}{2}\sin\phi\sin\theta, 0, 0, \frac{1}{\sqrt{2}}c(s) \right\} .
$$

3.2. *Identifying the interpolation functions*

Employing the Cartesian parameterization $v^\mu = \{w, x, y, z\}$ and Eq. (15) for the vector $\mathbf{p}_i(w, x, y, z)$, $i = 1 \ldots 11$ in $\mathbb{R}^{11}$, we can now compute the induced metric from the usual formula

$$
g_{\mu\nu} = \sum_{i=1}^{11} \frac{\partial \mathbf{p}_i(w, x, y, z)}{\partial v^\mu} \frac{\partial \mathbf{p}_i(w, x, y, z)}{\partial v^\nu} .
\tag{16}
$$

Using $r^2 = w^2 + x^2 + y^2 + z^2$ to simplify the notation, we find for the first column,

$$\frac{1}{2r^4}\begin{bmatrix} w^2\,r^2\left(a'(r)^2+b'(r)^2+c'(r)^2\right)+2\left(r^2-w^2\right)\left(a(r)^2+b(r)^2\right)-4z^2a(r)b(r) \\[2mm] -2wx\left(a(r)^2+b(r)^2\right)+4yz\,a(r)b(r)+wx\,r^2\left(a'(r)^2+b'(r)^2+c'(r)^2\right) \\[2mm] -2wy(a(r)^2+b(r)^2)-4xz\,a(r)b(r)+wy\,r^2\left(a'(r)^2+b'(r)^2+c'(r)^2\right) \\[2mm] wz\,r^2\left(a'(r)^2+b'(r)^2+c'(r)^2\right)-2wz(a(r)-b(r))^2 \end{bmatrix},$$

and the other three columns follow this pattern. We can already see the groupings of the unknown interpolation terms into factors identifiable with f, g, h in Eq. (6). The algebraic forms become clearer if we go to polar coordinates and compare with Eq. (8):

$$\begin{bmatrix} \frac{1}{2}\left(a'^2+b'^2+c'^2\right) & 0 & 0 & 0 \\[1mm] 0 & \frac{1}{4}\left(a^2+b^2\right) & 0 & 0 \\[1mm] 0 & 0 & \frac{1}{4}\left(a^2+b^2-2ab\cos^2\theta\right) & \frac{1}{4}\left(a-b\right)^2\cos\theta \\[1mm] 0 & 0 & \frac{1}{4}\left(a-b\right)^2\cos\theta & \frac{1}{4}\left(a-b\right)^2 \end{bmatrix} . \quad (17)$$

Collecting corresponding terms, we discover exactly three groups of the embedding interpolation functions $a(r)$, $b(r)$, and $c(r)$ and their first derivatives that correspond to $f(r)$, $g(r)$, and $h(r)$ in Eqs. (6) and (8),

$$a'(r)^2+b'(r)^2+c'(r)^2 \rightarrow 2f(r) \quad (18)$$

$$a(r)^2+b(r)^2 \rightarrow r^2g(r) \quad (19)$$

$$(a(r)-b(r))^2 \rightarrow r^2h(r) . \quad (20)$$

If we solve Eqs. (19) and (20) for $a(r)$ and $b(r)$ using Eq. (5), we find

$$a(r)=\frac{\sqrt{r^4+\sqrt{r^8-s^8}}}{\sqrt{2}r} \quad (21)$$

$$b(r)=\frac{s^4}{\sqrt{2}r\sqrt{r^4+\sqrt{r^8-s^8}}} , \quad (22)$$

with $a(s)=s/\sqrt{2}$, $a(r\rightarrow\infty)\rightarrow r$ and $b(s)=s/\sqrt{2}$, $b(r\rightarrow\infty)\rightarrow 0$. From Eqs. (18), (21), and (22), we can define $c(r)$ by its differential equation:

$$c'(r)=\sqrt{\frac{3s^4+r^4}{s^4+r^4}} . \quad (23)$$

We can then solve Eq. (23) for $c(r)$ with an interpolating function or use the explicit form

$$c(r)=\sqrt{3}rF_1\left(\frac{1}{4};\frac{1}{2},-\frac{1}{2};\frac{5}{4};-\frac{r^4}{s^4},-\frac{r^4}{3s^4}\right) , \quad (24)$$

where F_1 is the first Appell function. $c(r)$ has the properties:

$$c(s)=1.65069s$$

$$c(r\rightarrow\infty)=r+\text{const}(s)$$

$$c'(s)=\sqrt{2}$$

$$c'(\infty)=1 .$$

These same expressions can be obtained without reference to Eqs. (18, 19, 20), with some effort, by computing the Ricci tensor directly and solving the resulting differential equations for $a(r)$, $b(r)$, and $c(r)$.

With these results, Eq. (15) now defines a smooth map from $\mathbf{T}^*\mathbf{S}^2$ into $\mathbb{R}^{11}$ whose induced metric is Ricci-flat.

3.3. *Visual Representations of the* $\mathbb{R}^{11}$ *Embedding*

Now that we have explicit forms for the interpolation functions that create an isometric embedding of the Ricci-flat geometry with the topology of $\mathbf{T}^*\mathbf{S}^2$, we can examine the shapes of these functions. In Figure 1, we plot the forms of $a(r)$, $b(r)$, and $c(r)$ as well as $c'(r)$. We see that for the purposes of the embedding, the 11th dimension described by $c(r)$ is almost a straight line with unit slope, although it plays a *critical* role in the behavior of $f(r)$ near the origin to enforce the Ricci-flat condition in that neighborhood. Past a radius of about twice the radius s of the $\mathbf{S}^2$ at the origin, the shape of the constant-r cross-section is already essentially in the asymptotic form of a canonical $\mathbb{R}\mathbf{P}^3$ corresponding to the quadratic Veronese map of the underlying $\mathbf{S}^3$. This $\mathbb{R}\mathbf{P}^3$ embodies the ALE property of our manifold.

Next, we present some 2D cross-sections of our isometrically embedded 4-manifold projected somewhat arbitrarily from $\mathbb{R}^{11}$ to 3D to give an impression of the shape. In Figure 2a, we show a cutaway of surfaces sampled in r moving out from the $\mathbf{S}^2$ "core" at $r = s$. The full shape that is swept out becomes quite complex even with r sampled near to s, as shown in Figure 2b. Figure 3a picks a selection of latitude-longitude samples *on the surface* of $\mathbf{S}^2$, and shows the disks formed by sweeping out a segment in r in the collapsed Hopf variable ψ (see Eqs. (7) and (14)). As the maximum disk radius r moves outward from the $\mathbf{S}^2$ surface, we see in Figure 3b that the boundary circles of the disks begin to delineate a sampling of the ALE boundary 3-manifold $\mathbb{R}\mathbf{P}^3$ parameterized by ψ and the $\mathbf{S}^2$ variables θ and ϕ.

Finally, in Figure 4a, we present just the sampled $\mathbb{R}\mathbf{P}^3$ described by the boundary circles for large radius r with origin at sampled latitude-longitude pairs on the $\mathbf{S}^2$. These are essentially $\mathbb{Z}_2$ identifications of the more familiar Hopf fiber rings embedded in $\mathbf{S}^3$, which we show in Figure 4b for comparison. All this follows from the fact that our Veronese map of Eq. (12) is quadratic in the coordinates of $\mathbf{S}^3$, and hence double covered in $\mathbb{R}\mathbf{P}^3$, so each boundary circle in Figure 4a corresponds in parameter space to one half of the corresponding circle in Figure 4b.

4. A Direct Hyperkähler Quotient Embedding for A_1

We next turn to an alternative method of embedding a Ricci-flat metric for the A_1 Einstein space that depends on the hyperkähler constructions, but has a much less obvious geometric origin than the one we have presented in Section 3.

The hyperkähler moment map constraints for A_1 are expressed by Hitchin[14] in

terms of a pair of 2D complex variables, $\mathbf{z} = \{z_1, z_2\} \in \mathbb{C}^2$ and $\mathbf{w} = \{w_1, w_2\} \in \mathbb{C}^2$ obeying

$$\mu_1 = \|\mathbf{z}\|^2 - \|\mathbf{w}\|^2 = 1 \tag{25}$$

$$\mu_c = \mathbf{z} \cdot \mathbf{w} = 0 \ . \tag{26}$$

Note that these constraints are *invariant* under a $\mathbf{U}(1)$ phase transformation

$$\mathbf{z} \to e^{+\mathrm{i}\phi}\mathbf{z} \tag{27}$$

$$\mathbf{w} \to e^{-\mathrm{i}\phi}\mathbf{w} \ . \tag{28}$$

The construction of a metric starts with the eight real variables of $\{\mathbf{z}, \mathbf{w}\}$ and applies the three real constraints (one real, one complex) of Eqs. (25) and (26), thus defining an embedding of a five-manifold in $\mathbb{R}^8$. However, from Eqs. (27) and (28) it is clear that if the phase ϕ is included properly as the fifth variable, we can perform a quotient in ϕ, and, according to hyperkähler theory, the induced metric in 5D will then collapse in the quotient to the desired Ricci-flat metric on the A_1 4-manifold $\mathbf{T}^*\mathbf{S}^2$.

4.1. *Ansatz for the A_1 hyperkähler quotient*

We will proceed to solve the constraint equations in two steps, allowing us to expose some interesting details. We note that this process appears to be straightforward for A_1, but that at this time we have discovered no similar approach to solving A_k with $k > 1$; although there are alternative quotienting procedures in the twistor literature[8] that can in principle produce the A_k solutions, these are very cumbersome for $k > 1$. However, there do exist related twistor methods that convert the nonlinear Monge-Ampère equations into a linearized Laplacian-based Gibbons-Hawking system that has solutions for all k.[7,8]

By inspection, we can find a parametric solution of both moment map constraints Eqs. (25) and (26) in $\mathbb{C}^4$ ($\mathbb{R}^8$) with a combination of trigonometric and hyperbolic expressions in four real variables $\{x, t, a, b\}$ as follows:

$$z^0_{\ 1} = e^{ia} \cos(t) \cosh(x)$$
$$z^0_{\ 2} = e^{ib} \sin(t) \cosh(x)$$
$$w^0_{\ 1} = -e^{ib} \sin(t) \sinh(x)$$
$$w^0_{\ 2} = e^{ia} \cos(t) \sinh(x) \ . \tag{29}$$

If we compute the induced metric and curvature of the 4-manifold embedded in $\mathbb{R}^8$ defined by Eq. (29) and parameterized by $\{x, t, a, b\}$, we find that it is *not* Ricci-flat (in fact, there is exactly one non-zero Ricci component, $R_{11} = 6/\cosh^2(2x)$). But of course we still have the invariance under Eqs. (27) and (28), and so we can extend this to a 5D manifold that, due to the opposite phases of the transformations

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of $\mathbf{z}$ and $\mathbf{w}$, is distinct from multiplication by a pure global phase:

$$
\begin{aligned}
z_1 &= e^{\mathrm{i}(a+\phi)} \cos(t) \cosh(x) \\
z_2 &= e^{\mathrm{i}(b+\phi)} \sin(t) \cosh(x) \\
w_1 &= -e^{\mathrm{i}(b-\phi)} \sin(t) \sinh(x) \\
w_2 &= e^{\mathrm{i}(a-\phi)} \cos(t) \sinh(x) \ .
\end{aligned}
\tag{30}
$$

Considering $\mathbf{z}$ and $\mathbf{w}$ as a Euclidean space of eight parameterized functions $f_i(x,t,a,b,\phi)$, $i = 1\ldots8$ in $\mathbb{R}^8$, we can compute the induced metric on the 5D submanifold with parameters $v^\mu = \{x,t,a,b,\phi\}$ as

$$
g_{\mu\nu} = \sum_{i=1}^{8} \frac{\partial f_i(x,t,a,b,\phi)}{\partial v^\mu} \ \frac{\partial f_i(x,t,a,b,\phi)}{\partial v^\nu} \ .
\tag{31}
$$

The initial 5-dimensional induced metric with no quotienting is

$$
g_{\mu\nu} =
\begin{bmatrix}
\cosh(2x) & 0 & 0 & 0 & 0 \\
0 & \cosh(2x) & 0 & 0 & 0 \\
0 & 0 & \cos^2(t)\cosh(2x) & 0 & \cos^2(t) \\
0 & 0 & 0 & \sin^2(t)\cosh(2x) & \sin^2(t) \\
0 & 0 & \cos^2(t) & \sin^2(t) & \cosh(2x)
\end{bmatrix} \ .
\tag{32}
$$

While this again is not Ricci-flat, the 4×4 Ricci-flat metric follows from the quotient formula that eliminates the 5th gauge variable ϕ by projection,

$$
h_{ij} = g_{ij} - \frac{1}{g_{55}} g_{i5} g_{j5} \ .
\tag{33}
$$

The final result in coordinates $\{x,t,a,b\}$ becomes

$$
h_{ij} =
\begin{bmatrix}
\cosh(2x) & 0 & 0 & 0 \\
0 & \cosh(2x) & 0 & 0 \\
0 & 0 & \frac{\cos^2(t)\cosh^2(2x)-\cos^4(t)}{\cosh(2x)} & -\frac{\cos^2(t)\sin^2(t)}{\cosh(2x)} \\
0 & 0 & -\frac{\cos^2(t)\sin^2(t)}{\cosh(2x)} & \frac{(\cos(2t)+\cosh(4x))\sin^2(t)}{2\cosh(2x)}
\end{bmatrix} \ .
\tag{34}
$$

Direct computation verifies that the Riemannian curvature is nontrivial but the Ricci tensor vanishes identically,

$$
R_{ij}\left(h_{ij}\right) = 0 \ .
$$

This is of course known to be true from the hyperkähler quotient by construction, but nevertheless any given instance of a Ricci flat hyperkähler quotient metric is not necessarily trivial to express explicitly. In fact, at this writing, no ADE metrics have been successfully computed by any means, including explicit quotienting, except members of the A_k series.

5. Embedding Corresponding to A_1 Kähler Potential

In Section 4, we presented an explicit parametric solution of the $\mathbb{C}^4$ moment map constraints for the $k = 1$ gravitational instanton system combined with a $\mathbf{U}(1)$ quotient down to a 4D Ricci-flat metric. However, it turns out that there is a way to *avoid*, or, at least, sidestep, the explicit quotient process that we employed. One example using holomorphic coordinates and Kähler potential methods was known as early as 1980 in the physics supersymmetry literature[a] from the work of Alvarez-Gaumé and Freedman (AGF).[15] Here we include an outline of this alternative moment map constraint solution worked out in holomorphic coordinates to produce a Kähler potential, thus exposing another aspect of our story about ways to exploit embeddings to calculate the metric on $\mathbf{T}^*\mathbf{S}^2$. (We remark that there is in fact yet another Kähler potential for the $\mathbf{T}^*\mathbf{S}^2$ metric given by Gibbons and Pope[17] that produces a metric essentially in the coordinates used by Eguchi and Hanson; further details about the origins and form of that potential can be found in Lindström and Roček.[7])

The fields of the non-linear σ model in the AGF formulation are the four complex (8 real) coordinates $\{\mathbf{z}, \mathbf{w}\}$ in our approach, and the starting point of their parameterization is the homogeneous coordinate system $\mathbf{U} = \{u_0, u_1\}$ on $\mathbb{CP}^1$. A solution of the hyperkähler moment map constraints can be extracted starting with a complex normalized coordinate system (resembling the approach one would take to construct the Kähler potential for a Fubini-Study metric) with an overall weight factor $f(u, v, \bar{u}, \bar{v})$:

$$\mathbf{z} = f(u, v, \bar{u}, \bar{v}) \frac{\mathbf{U}}{\|\mathbf{U}\|} \quad = \quad f(u, v, \bar{u}, \bar{v}) \frac{\{1, u\}}{\sqrt{(1 + u\bar{u})}} \tag{35}$$

$$\mathbf{w} = f(u, v, \bar{u}, \bar{v}) \frac{\mathbf{V}}{\|\mathbf{V}\|} \quad = \quad f(u, v, \bar{u}, \bar{v}) \frac{\{1, v\}}{\sqrt{(1 + v\bar{v})}} \ , \tag{36}$$

where we scale from homogeneous to inhomogeneous coordinates on $\mathbb{CP}^1$, and note that it is essential for the complex 2-vector $\mathbf{z}$ to transform under the gauge action as $\exp(+\mathrm{i}\theta)$ while the complex 2-vector $\mathbf{w}$ transforms as $\exp(-\mathrm{i}\theta)$. We see that these definitions of $\mathbf{z}(u, v, \bar{u}, \bar{v})$ and $\mathbf{w}(u, v, \bar{u}, \bar{v})$ facilitate satisfying the moment map constraints in the form of Eqs. (25) and (26), but with a *distinct* set of constants,

$$\mu_1 = \|\mathbf{z}\|^2 - \|\mathbf{w}\|^2 = 0 \tag{37}$$

$$\mu_c = \mathbf{z} \cdot \mathbf{w} = 1 \ . \tag{38}$$

With the ansatz of Eqs. (35) and (36), we see that Eq. (37)) is satisfied trivially, and Eq. (38) is satisfied if

$$f(u, v, \bar{u}, \bar{v}) = \left[\frac{(1 + u\bar{u})(1 + v\bar{v})}{(1 + uv)^2} \right]^{1/4} \ . \tag{39}$$

---

[a]We thank Martin Roček for pointing this out.

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Although the constraint solution following from Eqs. (37) and (38) with the weight function Eq. (39) is not an isometric embedding, it leads directly to a metric through its associated Kähler potential,

$$K(u, v, \bar{u}, \bar{v}) = f(u, v, \bar{u}, \bar{v})\bar{f}(u, v, \bar{u}, \bar{v})$$

$$= \left(\frac{(1 + u\bar{u})(1 + v\bar{v})}{(1 + uv)(1 + \bar{u}\bar{v})}\right)^{1/2} . \qquad (40)$$

When we compute the Kähler metric $G = \partial\bar{\partial}K$ and take the log of the determinant, we find the Ricci potential $\Gamma = \log \det G$, from which one obtains the complex Ricci tensor via $R = \partial\bar{\partial}\Gamma$. The explicit value of Γ is easily computed to be

$$\Gamma(u, v, \bar{u}, \bar{v}) = -2\log(1 + uv) - 2\log(1 + \bar{u}\bar{v}) , \qquad (41)$$

so all mixed conjugation partials vanish, and thus G appears to be a Ricci flat metric on A_1. This is believed to yield a metric on $\mathbf{T}^*\mathbf{S}^2$ from general arguments presented in AGF,[15] but the authors did not demonstrate that explicitly. We observe that someplace in the process of working directly with the Kähler potential instead of the induced metric on the embedded constraint solution, the need to quotient out of the 5D embedding has disappeared, and the Ricci-flat Kähler potential has presented itself immediately in 4D holomorphic coordinates.

6. Concluding Remarks

The metrics on the ADE manifolds have been extensively studied since Hitchin[6] originally introduced the concept that all the possible 4D asymptotically locally Euclidean (ALE) self-dual solutions to Einstein's equations could be identified with the Kleinian groups, or, equivalently, to the corresponding lens spaces $\mathbf{S}^3/G$ of $\mathbf{S}^3$. At this moment, only the A_k metrics have explicit solutions, and embedding-based treatments are very difficult for $k > 1$. We have presented here three distinct ways of solving the $k = 1$ case, each giving Ricci-flat metrics on the topological space $\mathbf{T}^*\mathbf{S}^2$ of the A_1 gravitational instanton. Our first result was a novel approach based on embedding $\mathbb{R}\mathbf{P}^3$ in $\mathbb{R}^{11}$ and interpolating while continuously enforcing Ricci-flatness through to a Hopf fibration on the $\mathbf{S}^2$ "core" at the origin of $\mathbf{T}^*\mathbf{S}^2$, resulting in an isometric embedding giving precisely the Eguchi-Hanson form of the metric. Our second example was an explicit hyperkähler quotient of the induced metric of an embedded 5-manifold defined by a three parameter set of moment map constraints in a flat $\mathbb{R}^8$, which, combined with a $\mathbf{U}(1)$ quotient, reduced to a Ricci-flat four-dimensional metric. While this form of the metric is technically known, our presentation working out all the steps in detail seems useful. Finally, for completeness, we reviewed the the quotient-free constraint solution leading to the Kähler potential of Alvarez-Gaumé and Freedman, which provides yet another example of an embedding procedure that leads to a Ricci-flat metric for the A_1 gravitational instanton.
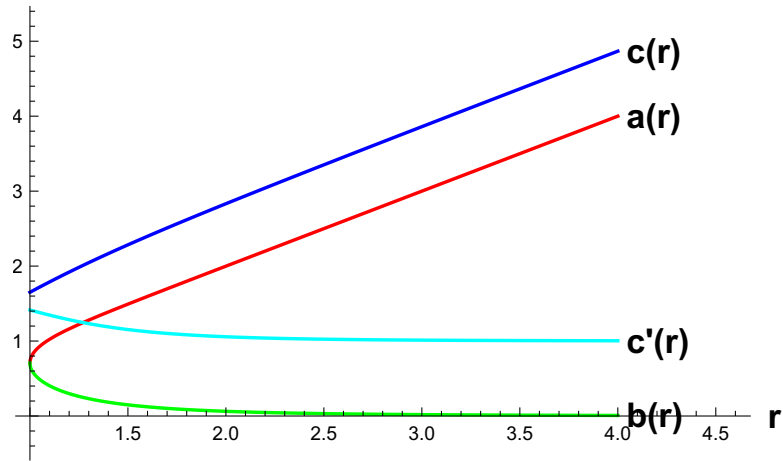
Fig. 1.   The interpolation functions for the Ricci-flat $\mathbb{R}^{11}$ isometric embedding of $\mathbf{T}^*\mathbf{S}^2$ for $\mathbf{S}^2$ radius $s = 1$. (red) $a(r)$; (green) $b(r)$; (blue) $c(r)$; (cyan) $c'(r)$.
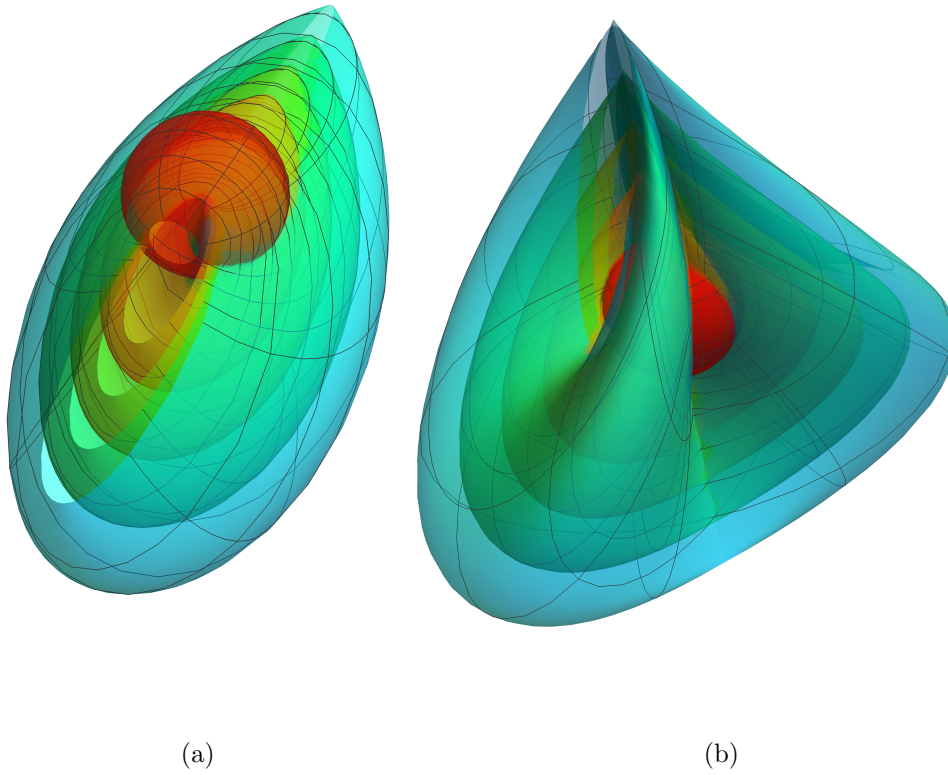

(a)   (b)

Fig. 2.   The $\mathbf{S}^2$ "core" at $r = s$ of the isometric embedding of $\mathbf{T}^*\mathbf{S}^2$ and its behavior for sampled values $s \leq r < 2s$. (a) View with several cutaway layers at sampled values with small r. (b) View of the entire layers deforming away from $\mathbf{S}^2$ at sampled r.

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(a)   (b)
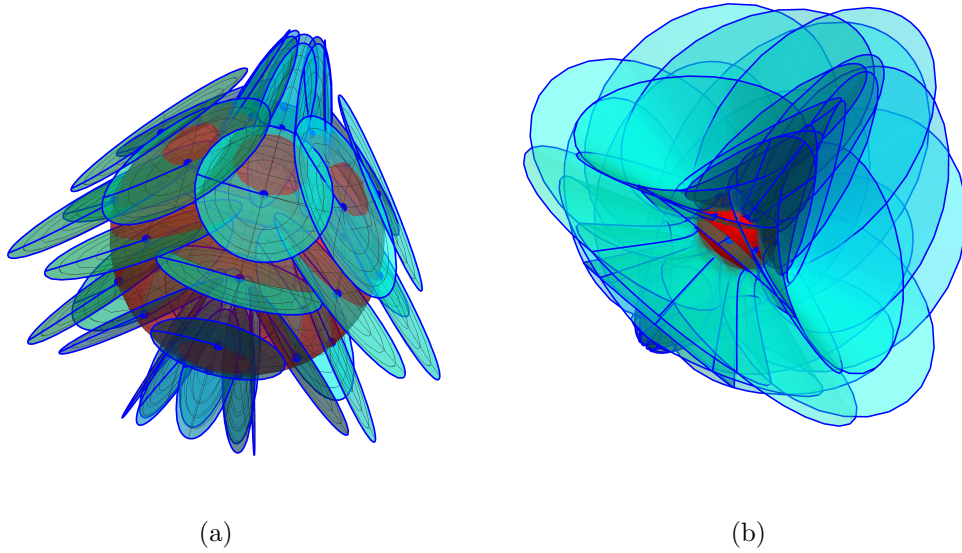
Fig. 3.   The A_1 4-manifold represented by disks in the radial and Hopf-fibration variables, sampled at values of the $\mathbf{S}^2$ latitude and longitude. (a) The nascent disks close to the $\mathbf{S}^2$ samples. (b) Expanding the disks away from the $\mathbf{S}^2$ "core" for larger values of sampled latitude/longitude, beginning to show the shape of $\mathbb{R}\mathbf{P}^3$ in the circles bounding the disks.

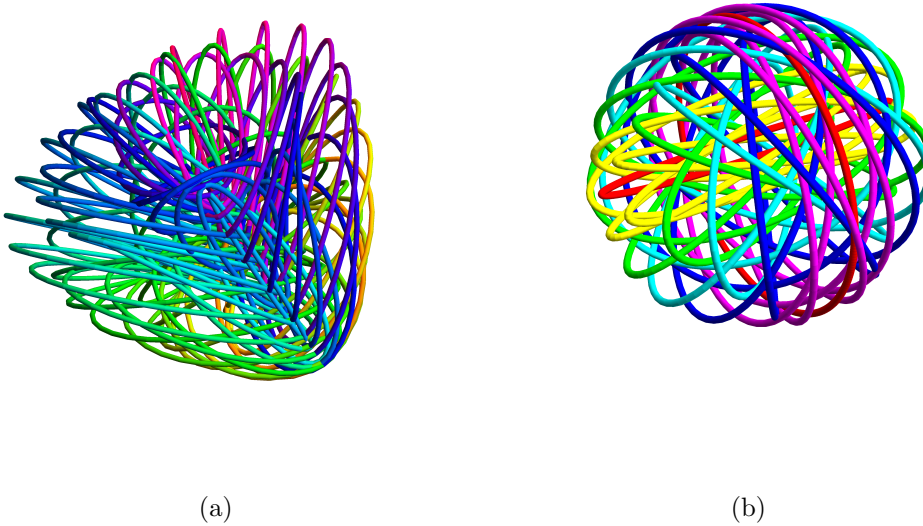(a)   (b)

Fig. 4.   (a) Far away from the sampled latitude-longitude points on $\mathbf{S}^2$, we tessellate the asymptotic $\mathbb{R}\mathbf{P}^3$ ALE boundary 3-manifold at a fixed large value of r, using rings in the fiber variable ψ that collapses and disappears on $\mathbf{S}^2$. (b) For comparison, these are the corresponding Hopf fiber rings tessellating $\mathbf{S}^3$ using similar $\mathbf{S}^2$ samples. These are essentially the double-covers of the rings in (a).

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