Illustrating String Theory Using Fermat Surfaces

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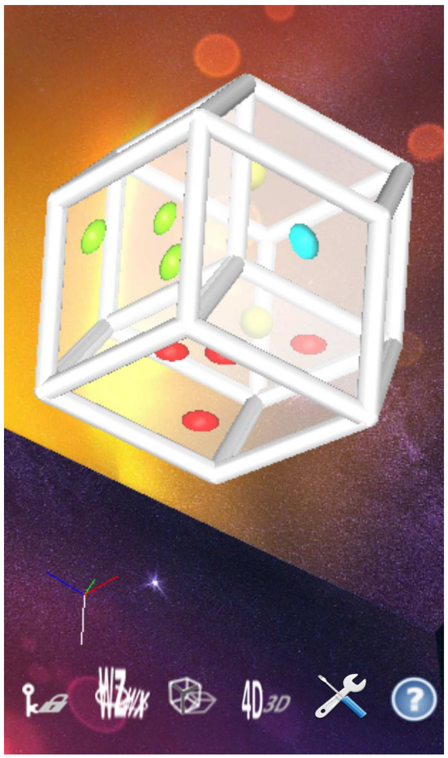
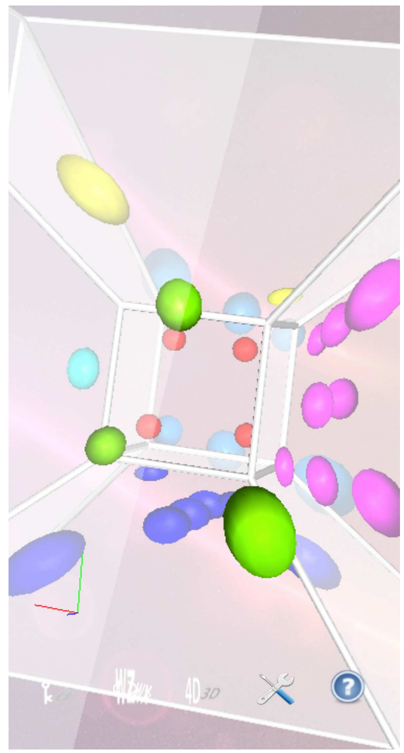
Illustrating Geometry and Topology, 16–21 Sept 2019

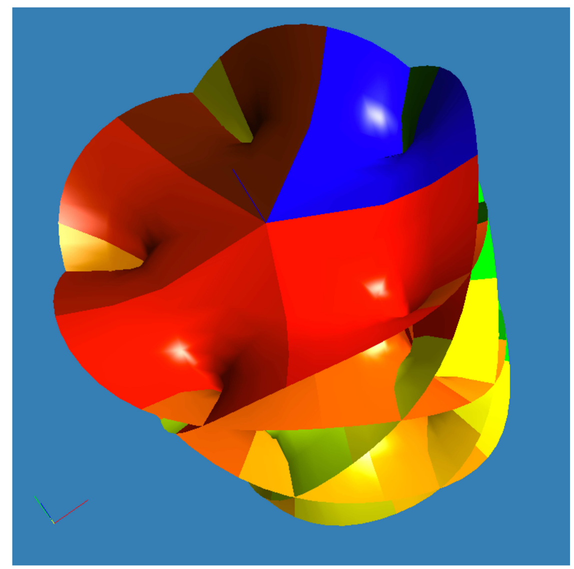
4D Intuition-Friendly User Interfaces:

4Dice | **4DRoom** | **4D Explorer**



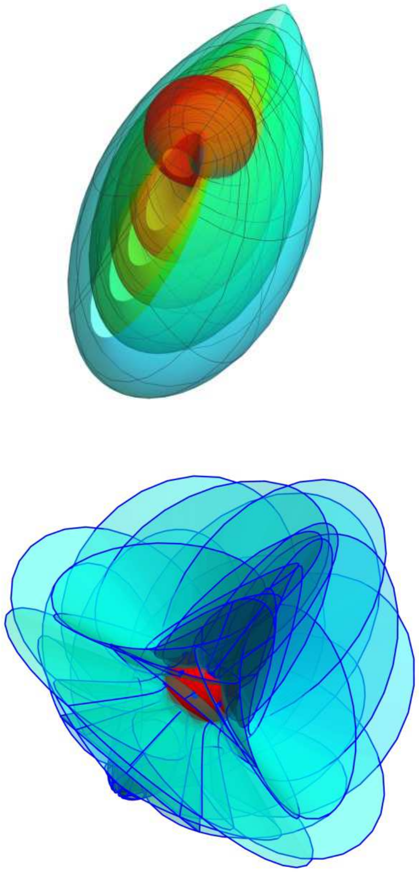


4D Explorer



Free on the App Store! || `http://homes.sice.indiana.edu/hansona`

Quaternion Proteomics || Isometric Einstein Embeddings



Quaternion applications to protein geometry and geometry-matching

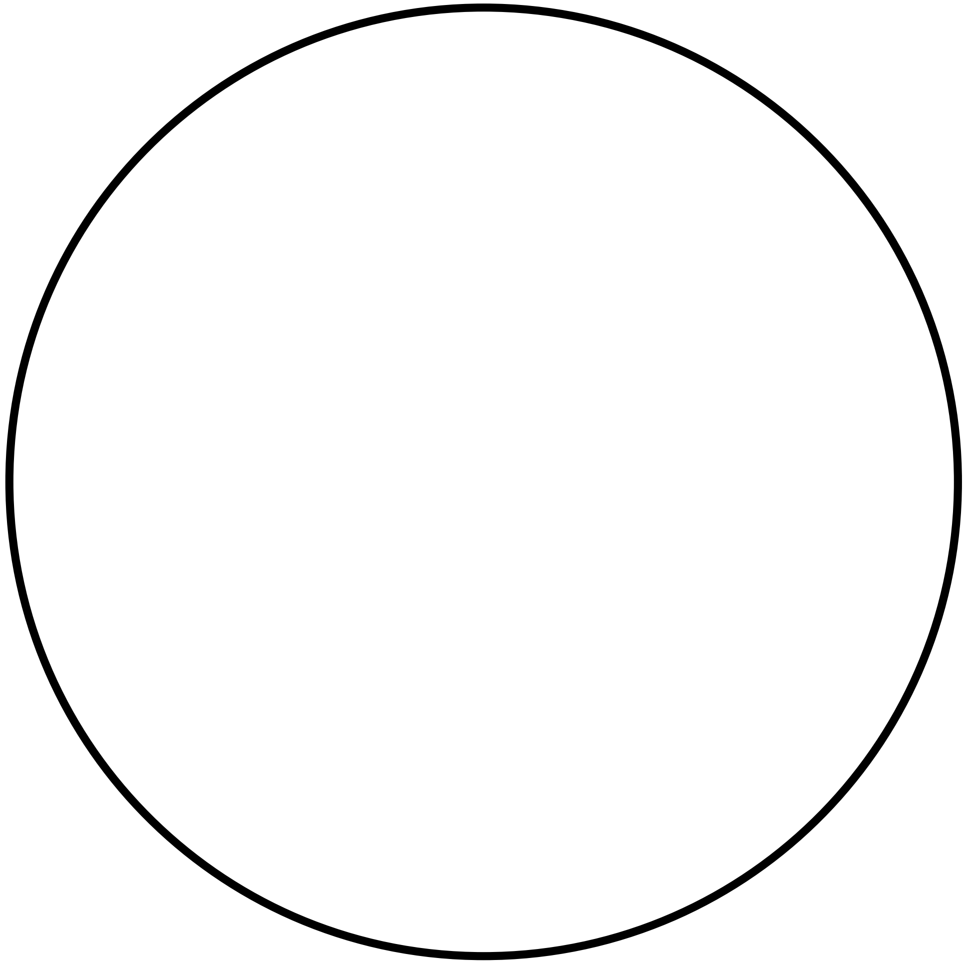
11D Nash embedding of self-dual Einstein metric

Onward to Fermat → Calabi-Yau

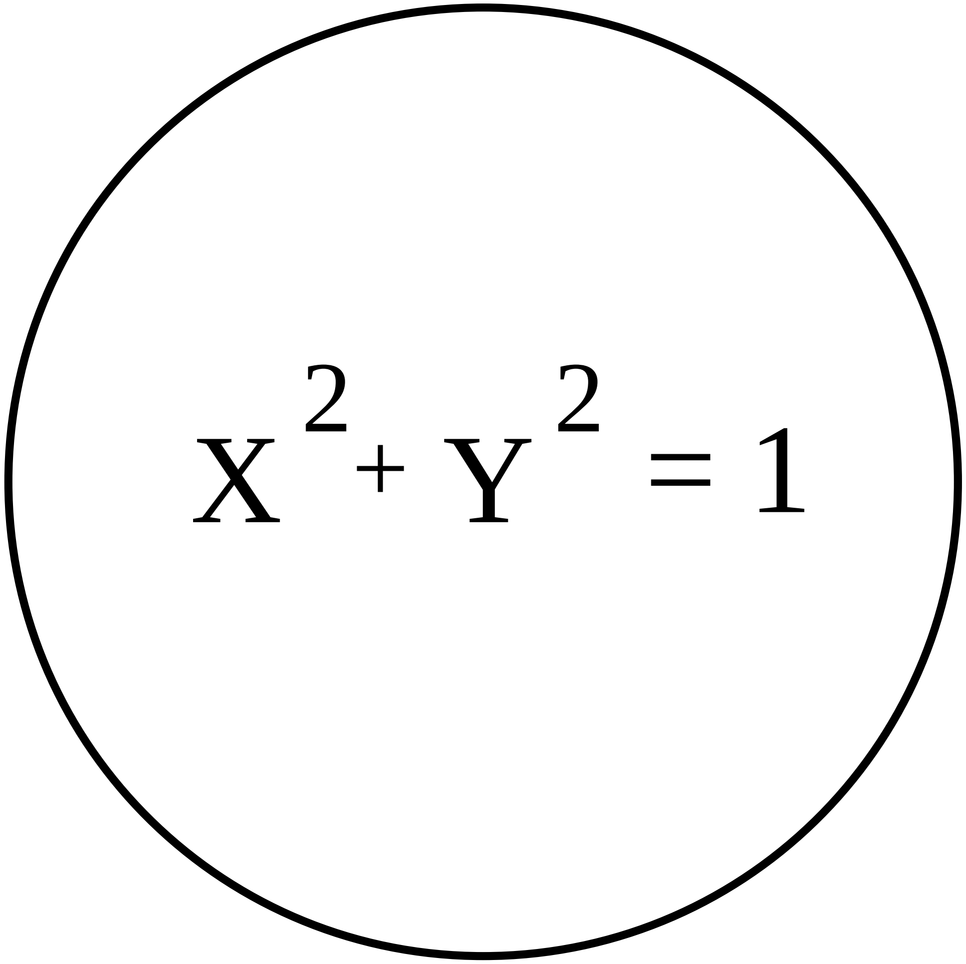
350 Years of a Common Thread:

- **(1637, 1995) Fermat's Last Theorem...**
- **(1959, 1981) Superquadrics...**
- **(1954, 1978, 1985) Calabi-Yau Spaces in String Theory...**
- **We will now *connect all these together*...**

The Common Thread Is This:



Implicit Equation of a Circle



$$X^2 + Y^2 = 1$$

...and its Parametric Trigonometric Solution:

$$X^2 + Y^2 = 1$$

$$X = \cos\theta \qquad Y = \sin\theta$$

Why a circle?

- Fermat's theorem involves changing the circle equation to *any integer power*.
- Superquadrics map the $(\cos\theta, \sin\theta)$ solutions to solve a circle-like equation for *any real power*.
- Leading examples of Calabi-Yau spaces that may describe the hidden dimensions of String Theory are *complexified extensions* of Fermat's equations.
- **So in a real sense: ALL WE NEED TO UNDERSTAND IS THE EQUATION OF A CIRCLE.**

Pierre de Fermat

1601(?)–1665

1637 — Fermat's "Last Theorem"

- **Fermat's "Last Theorem" states that**

  $$\boxed{x^p + y^p = z^p}$$

  has no solutions in positive integers for integers $\boxed{\mathrm{p} > 2}$.

- In 1637, Fermat wrote a note in the margin of his copy of the *Arithmetica* of Diophantus, claiming to have a proof that he never recorded or mentioned thereafter.

Arithmeticorum Liber II. 61

IN QVAESTIONEM VII.

QVÆSTIO VIII.

OBSERVATIO DOMINI PETRI DE FERMAT.

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos & generaliter nullam in infinitum vltra quadratum potestatem in duos eiusdem nominis fas est diuidere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

QVÆSTIO IX.

Annotated copy of ***Arithmetica*** **of Diophantus, published by Fermat's son and including Fermat's margin notes, stating**

"I have a marvelous proof that this margin is too small to contain."

Fermat's "Theorem," contd.

- In 1995, **Andrew Wiles** and collaborators proved the theorem using the most modern techniques of elliptic curve theory, unknowable by Fermat, but it is unknown whether a more elementary proof exists.

- In 1990, *before the proof*, I made a brief film, "Visualizing Fermat's Theorem" that I will show you shortly.

Next: 1959 — Traffic Circles on Steroids

- Danish poet Piet Hein designs a non-circular shape for a traffic roundabout in Stockholm in **1959**, with $p = 2.5$ and $(a/b) = (6/5)$:

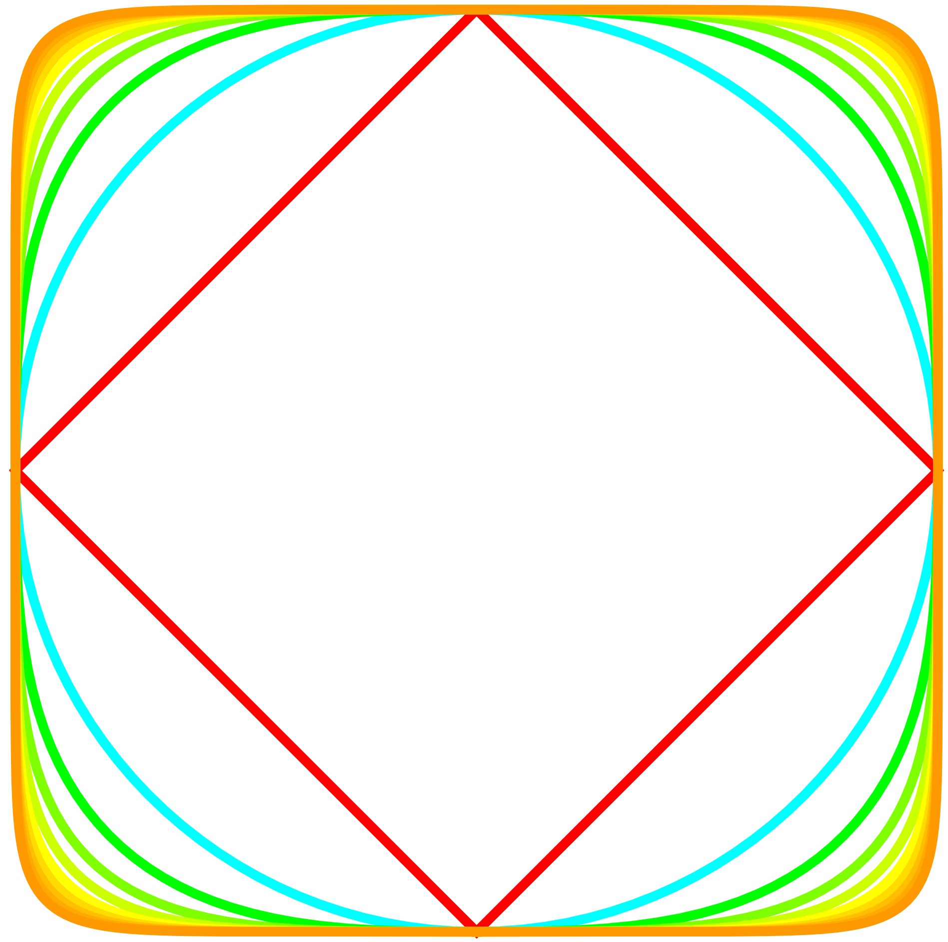
$$\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^p = 1$$

- Hein then popularized the **Super Egg** in 3D:

$$(az)^p + \left(b\sqrt{x^2 + y^2}\right)^p = 1$$

The Super Egg

The Super Circles



These are **"Real Fermat Curves"** *for integers from* $\boxed{p = 1 \ldots 10}$.
You may also recognize these as L_p **Norms.**

Footnote: The Super Fonts

Superquadrics may have actually entered the world first as *font design* parameters.

- **1952:** Herman Zapf's **Melior** type faces appear to have superquadric components.

- Donald Knuth's **Computer Modern** type faces explicitly contain superquadric shape design options.

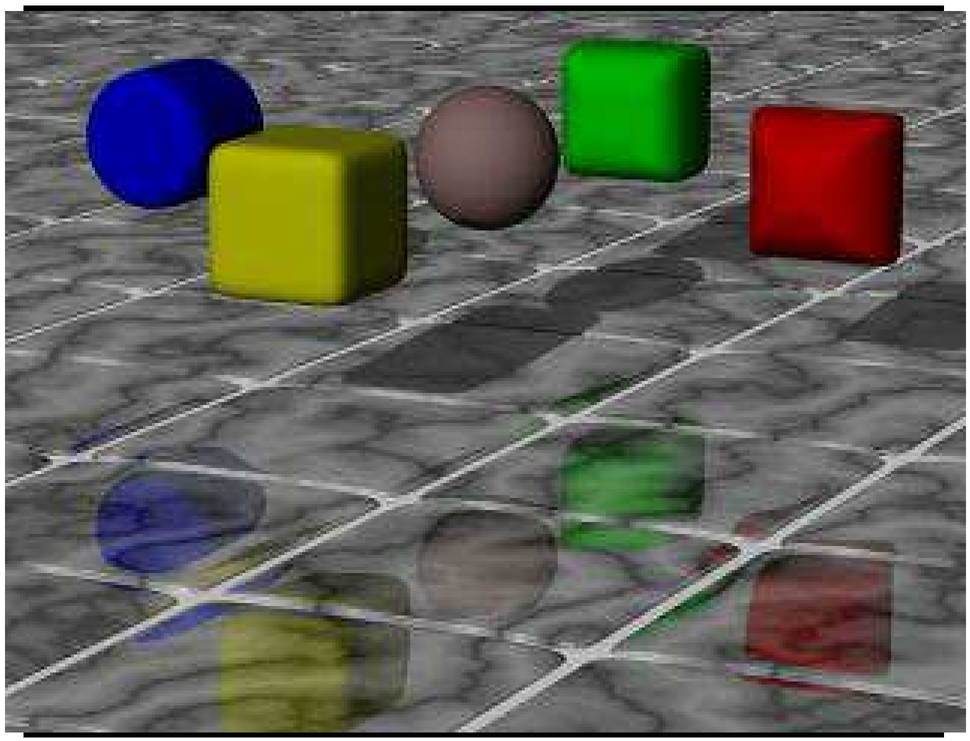
1981: Superquadrics meet Graphics

- Alan Barr introduces the class of Superquadric shapes to 3D computer graphics in the first issue of IEEE CG&A:

$$x^p + y^p + z^p = 1$$

- Many interesting tricks: exploit *continuously varying exponents* and ratios, invert equations for ray-tracing, toroidal variants, etc.

SuperQuadrics in POVRay

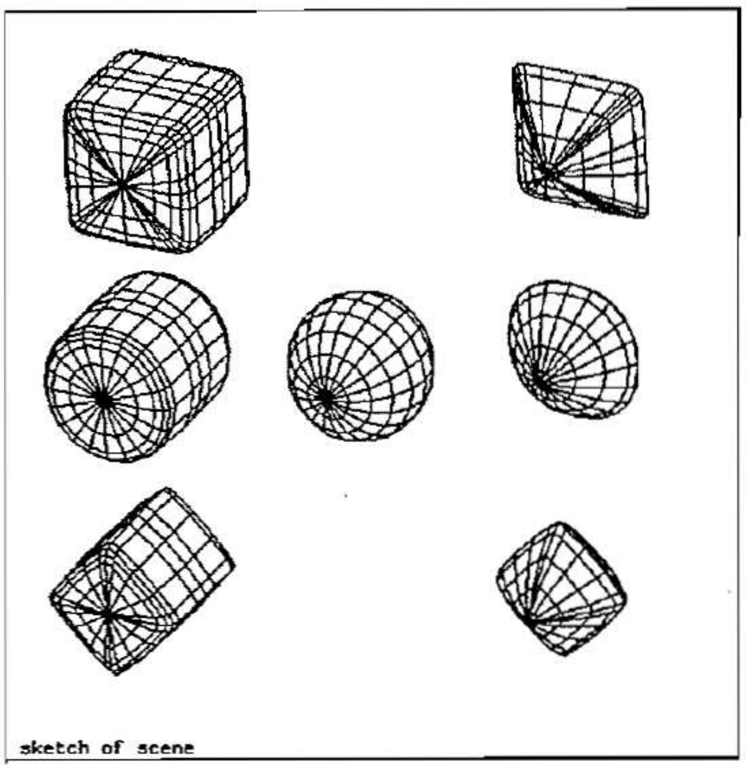


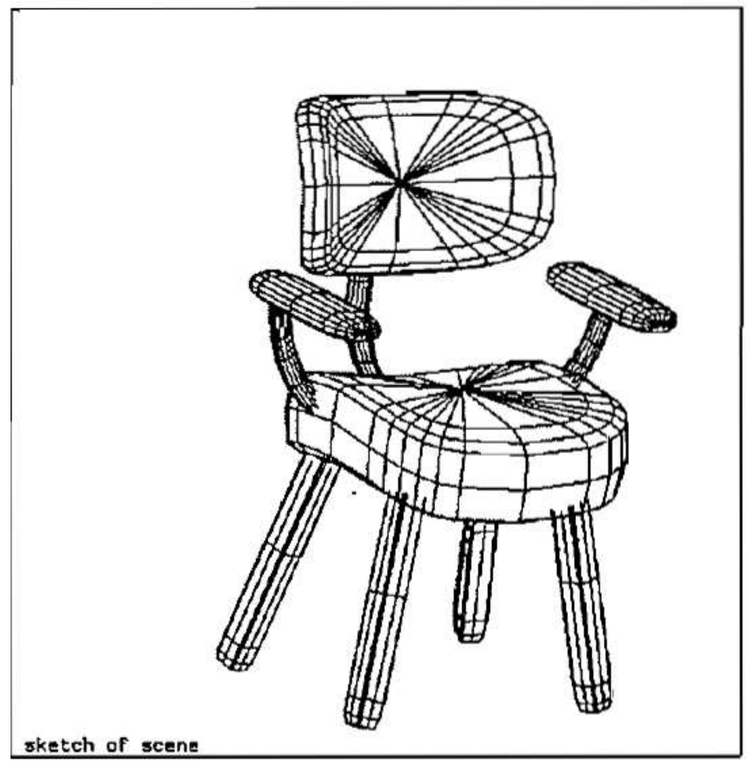
Superquadrics as primitives in popular graphics packages.

1987: Superquadrics Appear in Machine Vision

- **Alex Pentland** started using superquadrics as *shape recognition primitives*, and his ICCV '87 paper initiated a long literature.
- *Pentland*, who had the office next to mine at SRI in the mid 1980's, introduced me to Barr's paper and to superquadrics. . .
- **and that led me directly to notice the connection to Fermat's theorem...**

”SuperSketch” Quadric Shape Primitives





Superquadric/Fermat DEMO

Visualizing Superquadrics in a Fermat context

1990 — Fermat's Theorem Film

This film, focused on *Mathematical Visualization*, was shown first in 1990 at IEEE Visualization Conference in San Francisco, then the Siggraph 1990 Animation Festival.

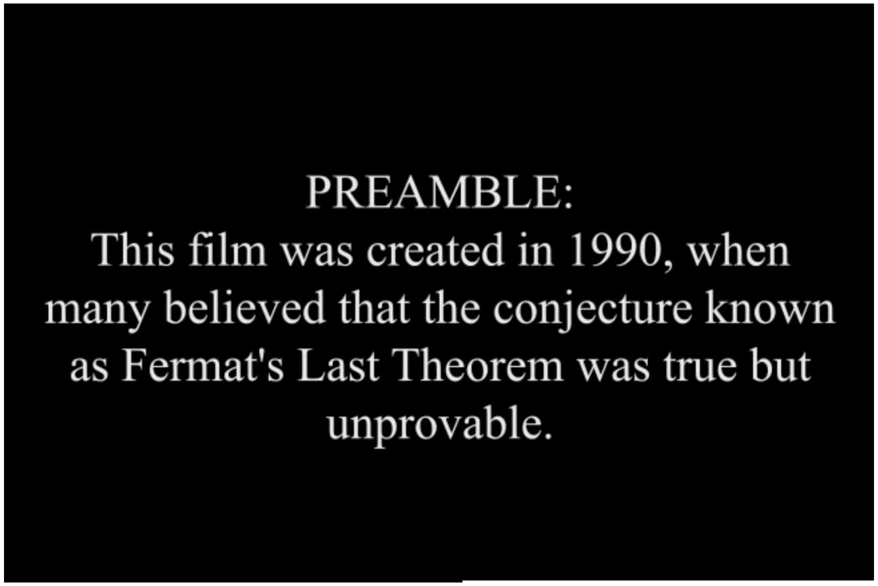
- **First:** I got involved in *Superquadrics*, and noted the resemblance to Fermat's "Theorem" equation:

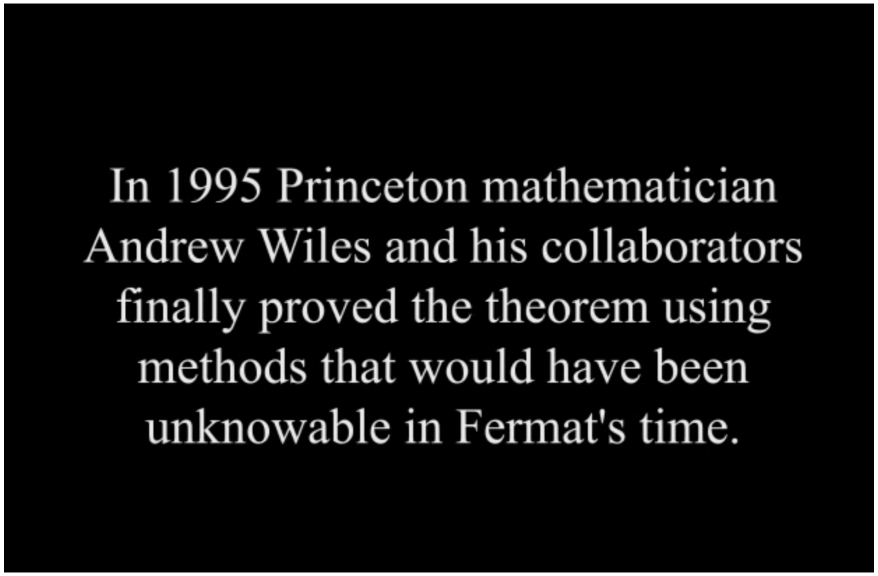
  $$(x/z)^p + (y/z)^p = 1$$

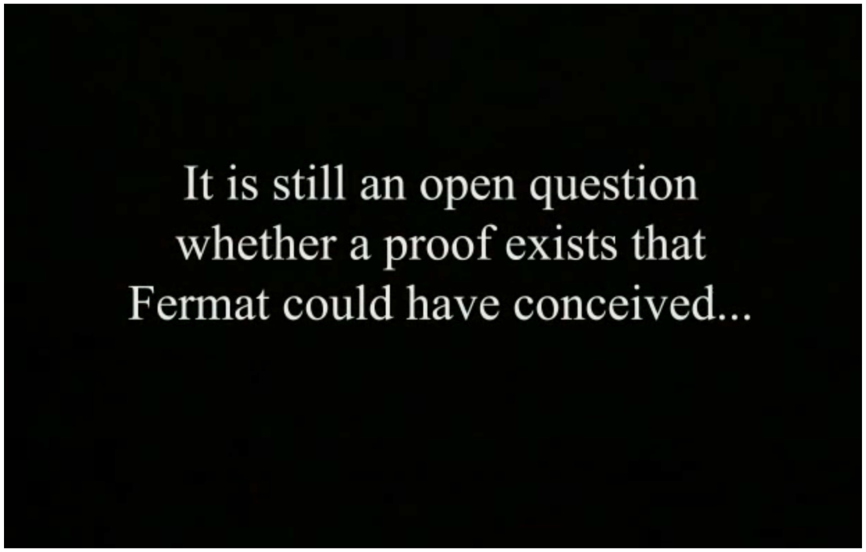
  which has no *rational* solutions for integers $p > 2$.

- **Then:** I asked John Ewing, an IU mathematician, if somehow the **superquadric graphics** might be useful to try to explain Fermat's theorem; he suggested *complexifying* the equation, leading to a **surface in 4D space**. (I found out *much later* that this was related to **Calabi-Yau spaces** and **string theory**, which we will discuss shortly.)

Preface to the film...







Fermat Film

Film: "Visualizing Fermat's Last Theorem"

`https://www.youtube.com/watch?v=xG63O031WZI`

"andjorhanson" YouTube channel

Apology: There was a tight time limit on short films submitted to the Siggraph '90 Animation Theater, and so this goes by *REALLY FAST*

Remember: This film was made years before Fermat's "theorem" was actually proven.

The String Theory Connection

- In the fall of 1998, I got a call from a physicist I'd never heard of named **Brian Greene**.

- **Somehow, he had come across my work on the visualization of Fermat surfaces, and thought they could be adapted for the figures showing** *Calabi-Yau Spaces* **in his forthcoming book on string theory** $\longrightarrow$ *The Elegant Universe*.

- **Somehow it all worked, and versions of those images have appeared in dozens of articles, etc., on string theory over the last two decades.**

What is a Calabi-Yau space?

- **Definition in a Nutshell**: A Calabi-Yau space is an N-complex-dimensional Kähler manifold with first Chern class $c_1 = 0$ and an identically vanishing Ricci tensor.
- **Calabi-Yau spaces are thus nontrivial solutions to the Euclidean vacuum Einstein equations**.
- ***This is as close to*** *flat* ***as you can get and still be nontrivial, which has very important potential applications.***

Why are people interested in CY spaces?

- **Physics:** Basic String Theory says spacetime is 10D; we only see 4D, so **6 Hidden Dimensions** are left — a Calabi-Yau Quintic in $\mathbb{CP}(4)$ works (though many other possibilities are now known).
- **Mathematics:** Mathematicians generally are happy with *EXISTENCE* proofs. But, though CY spaces with Ricci-flat metrics *EXIST*, no one has written down any solution. *A Major unsolved problem!*
- **Visualization:** If you can't write the metric down, *maybe "illustrating" CY spaces will help?*

The Simplest Calabi-Yau Manifolds

- $\mathbb{C}\mathbf{P}(N)$: The Calabi conjecture, proven by Yau, says the following manifold in $\mathbb{C}\mathbf{P}(N)$ admits a non-trivial Ricci-flat solution to Einstein's gravity equations:

  $$z_0{}^{N+1} + z_1{}^{N+1} + \cdots + z_N{}^{N+1} = 0$$

  E.g., $N = 2$ is a cubic embedded in $\mathbb{C}\mathbf{P}(2)$, which is simply a **torus** and admits a flat (thus Ricci-flat) metric.

- To get a 6-manifold, we need $N = 4$, implying a quintic polynomial embedded in $\mathbb{C}\mathbf{P}(4)$:

  $$z_0{}^5 + z_1{}^5 + z_2{}^5 + z_3{}^5 + z_4{}^5 = 0$$

Polynomial Calabi-Yau Manifolds, contd

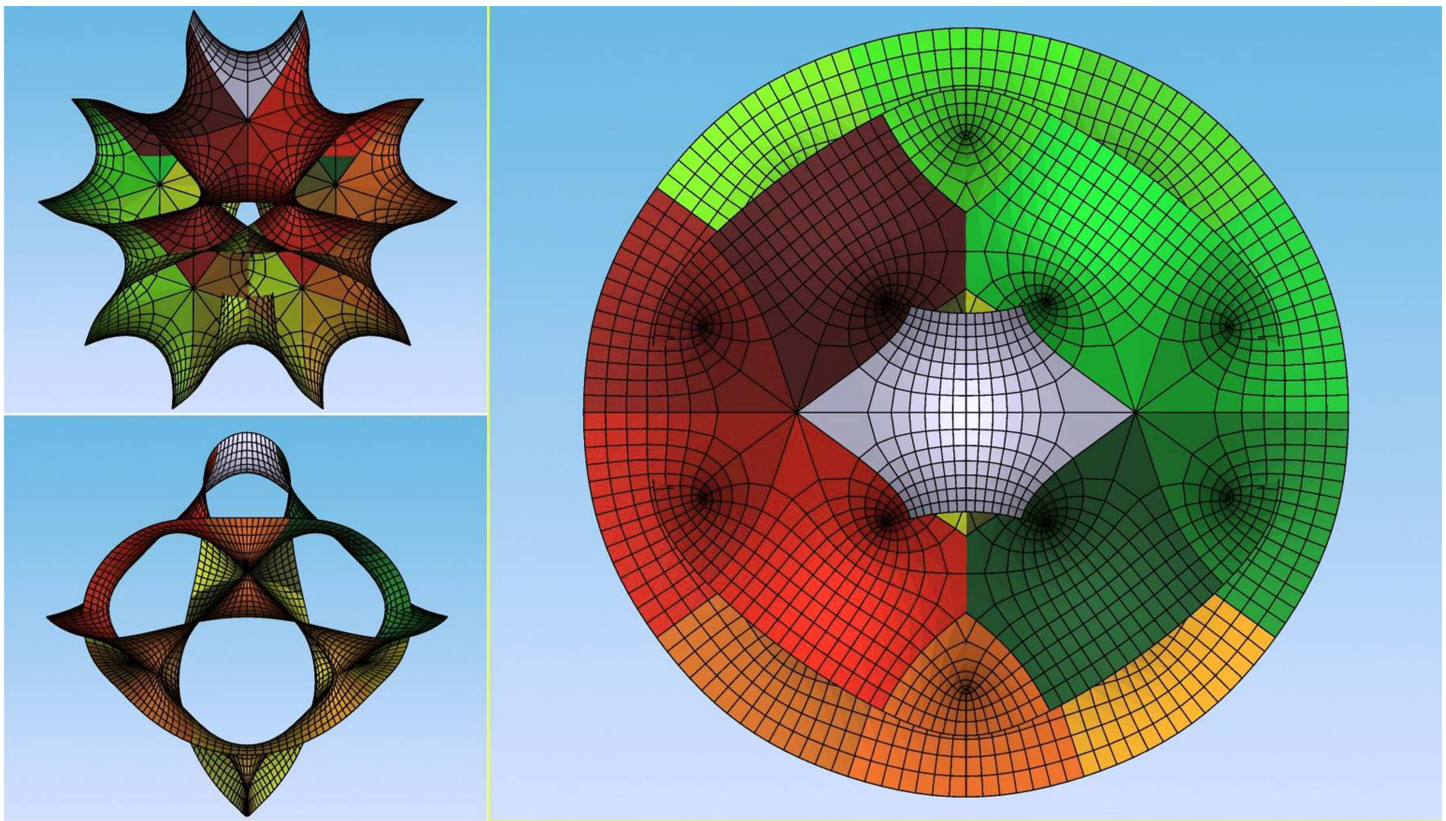
- For *any* $2(N-1)$-real-dimensional Calabi-Yau space in $\mathbb{C}\mathbf{P}(N)$, we can look at the 2-manifold cross-section in $\mathbb{C}\mathbf{P}(2)$, a 4D real space, by setting all the terms to constants except z_1 and z_2, and studying this 2D slice of the full space,

$$\boxed{{z_1}^{N+1} + {z_2}^{N+1} = 1\ ,}$$

and that is what we have done for $N = 4$, representing the quintic 6-manifold in $\mathbb{C}\mathbf{P}(4)$.

My 2D Cross-Section of the 6D Calabi-Yau Quintic:

Is this what the Six Hidden Dimensions look like?



Elegant Universe image of Calabi-Yau Quintic

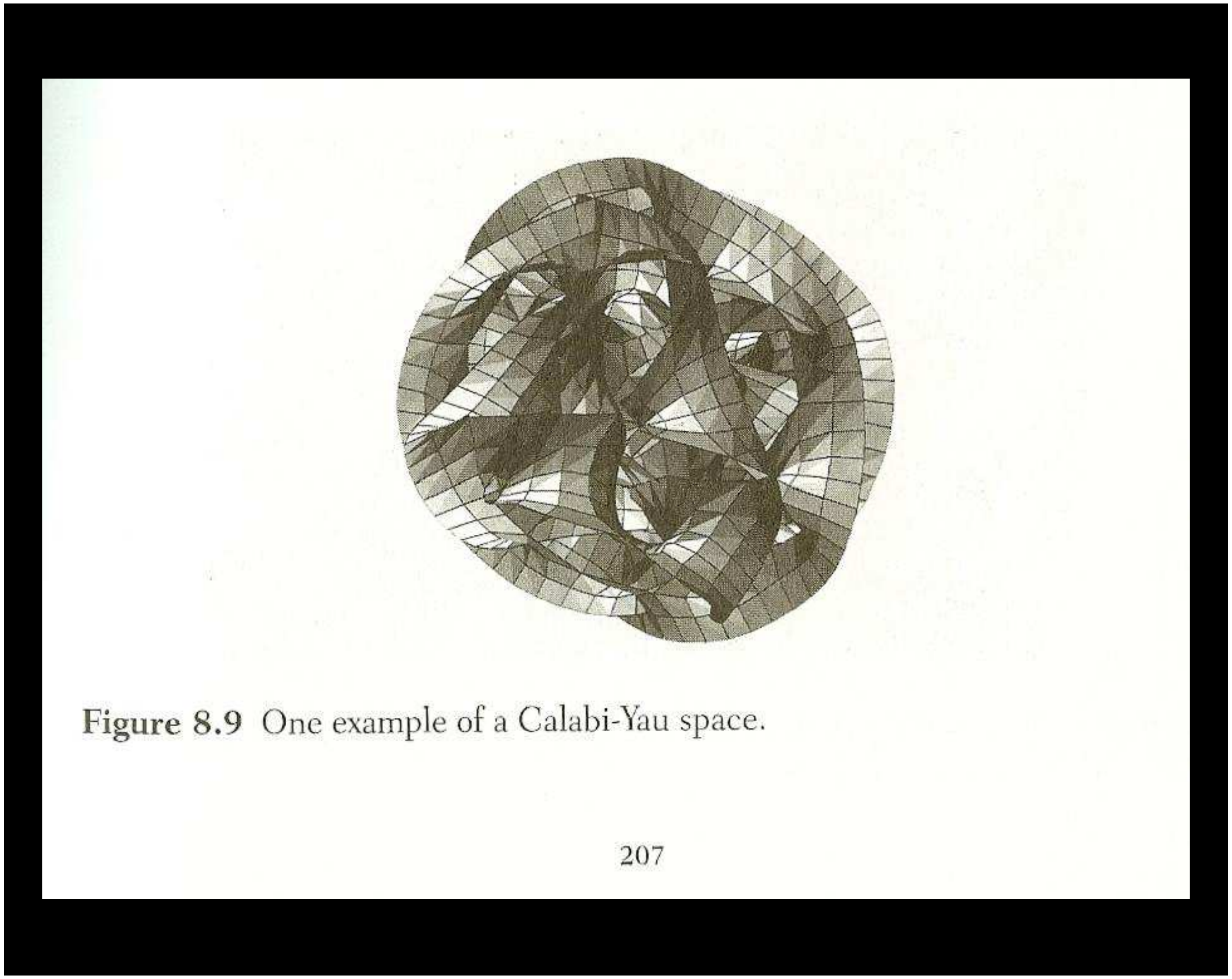


Figure 8.9 One example of a Calabi-Yau space.

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Elegant Universe GRID of Calabi-Yau Quintics

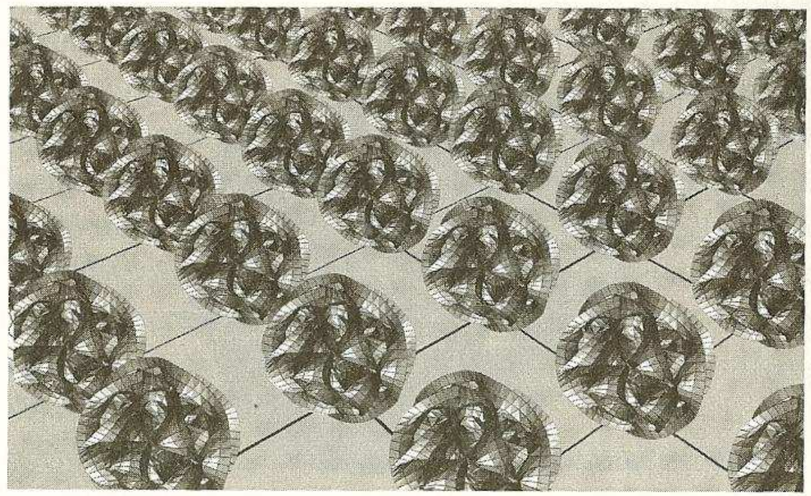

Figure 8.10 According to string theory, the universe has extra dimensions curled up into a Calabi-Yau shape.

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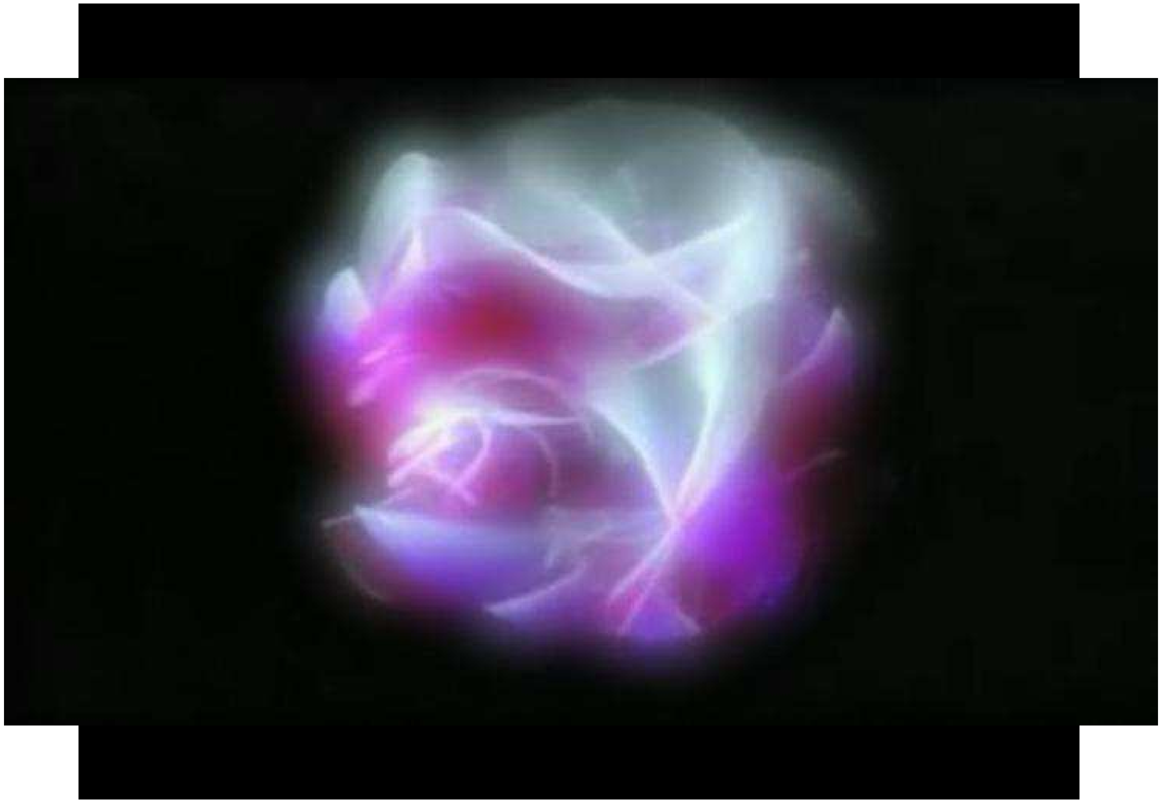
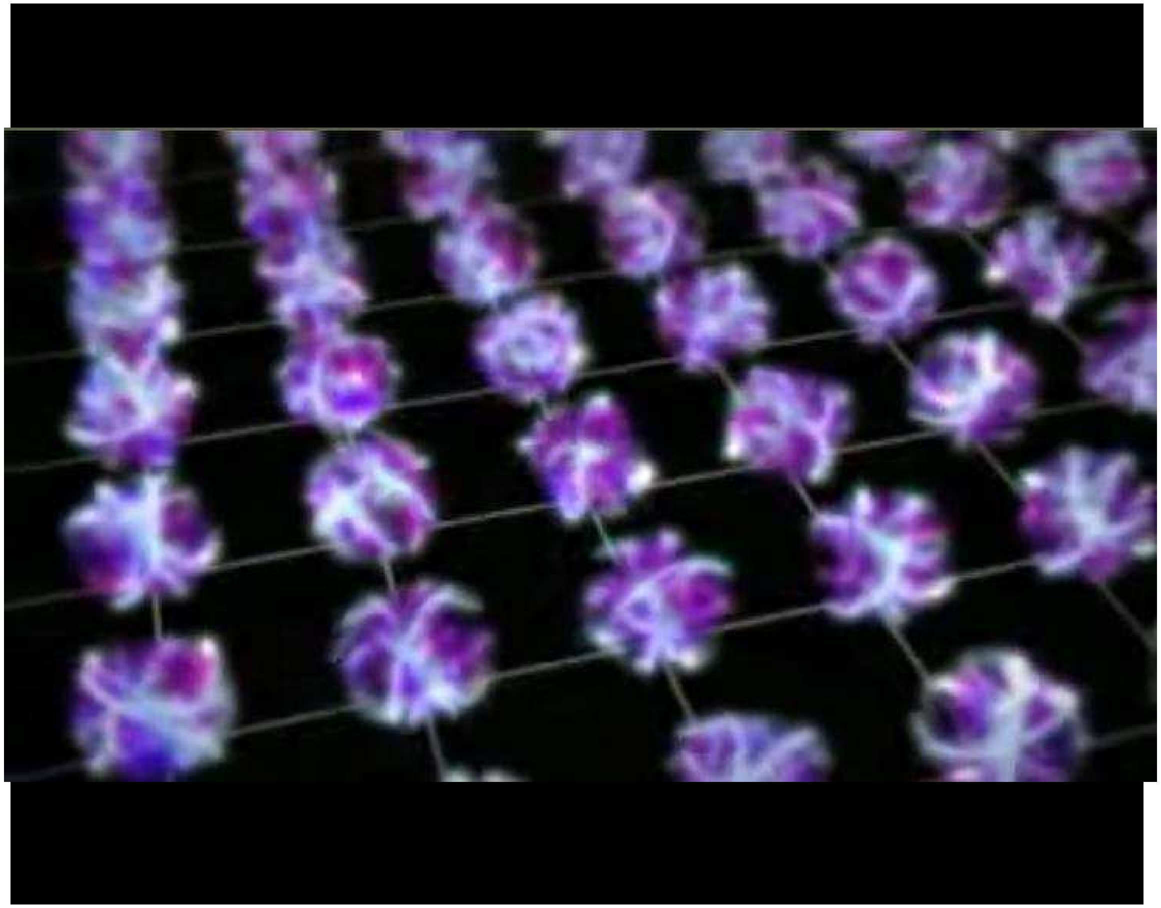
NOVA animations

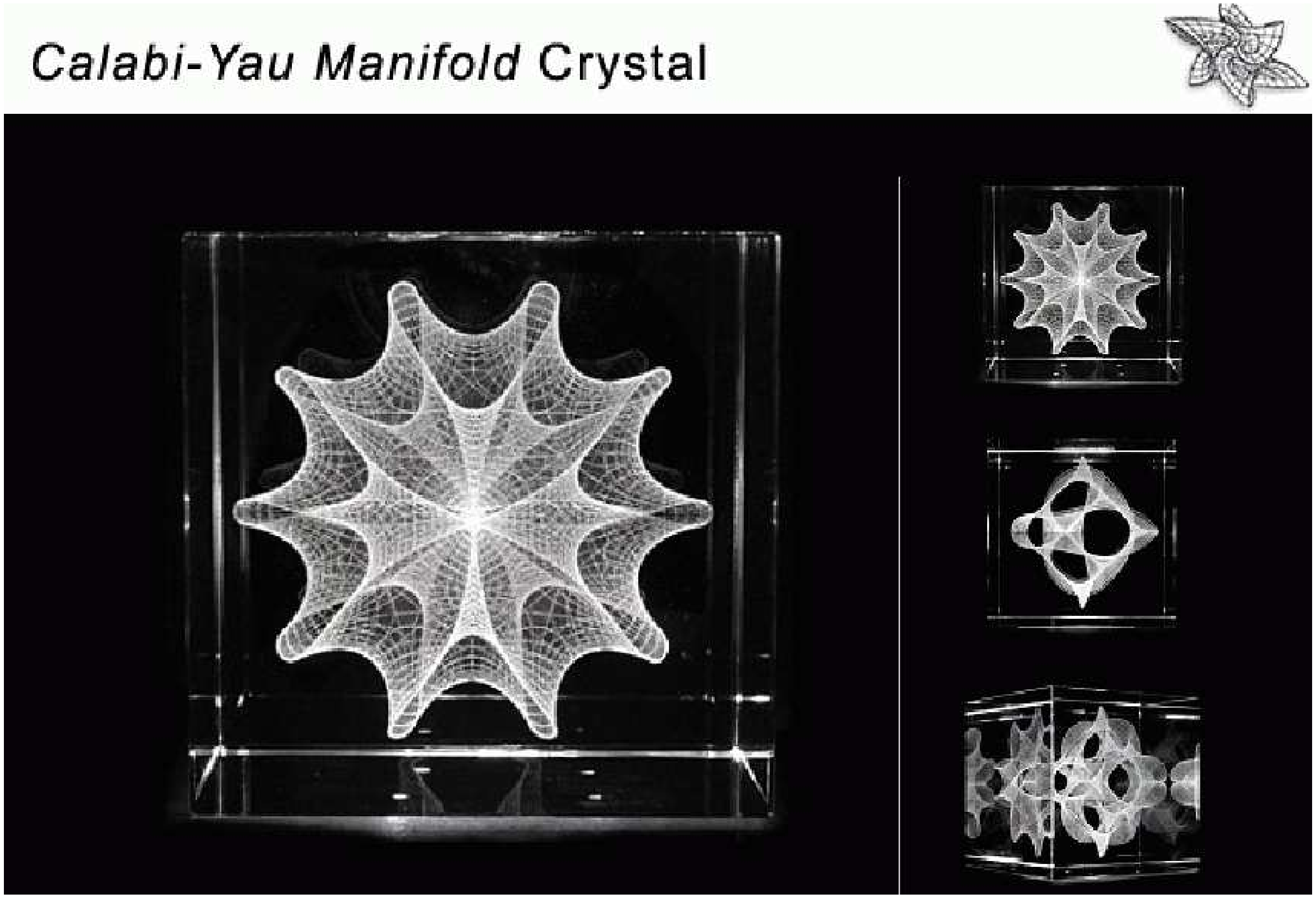
Greene's book led to a 3-part NOVA series on String Theory in the fall of 2003, with some fascinating professional animations:



NOVA *grid* of Calabi-Yau Quintic

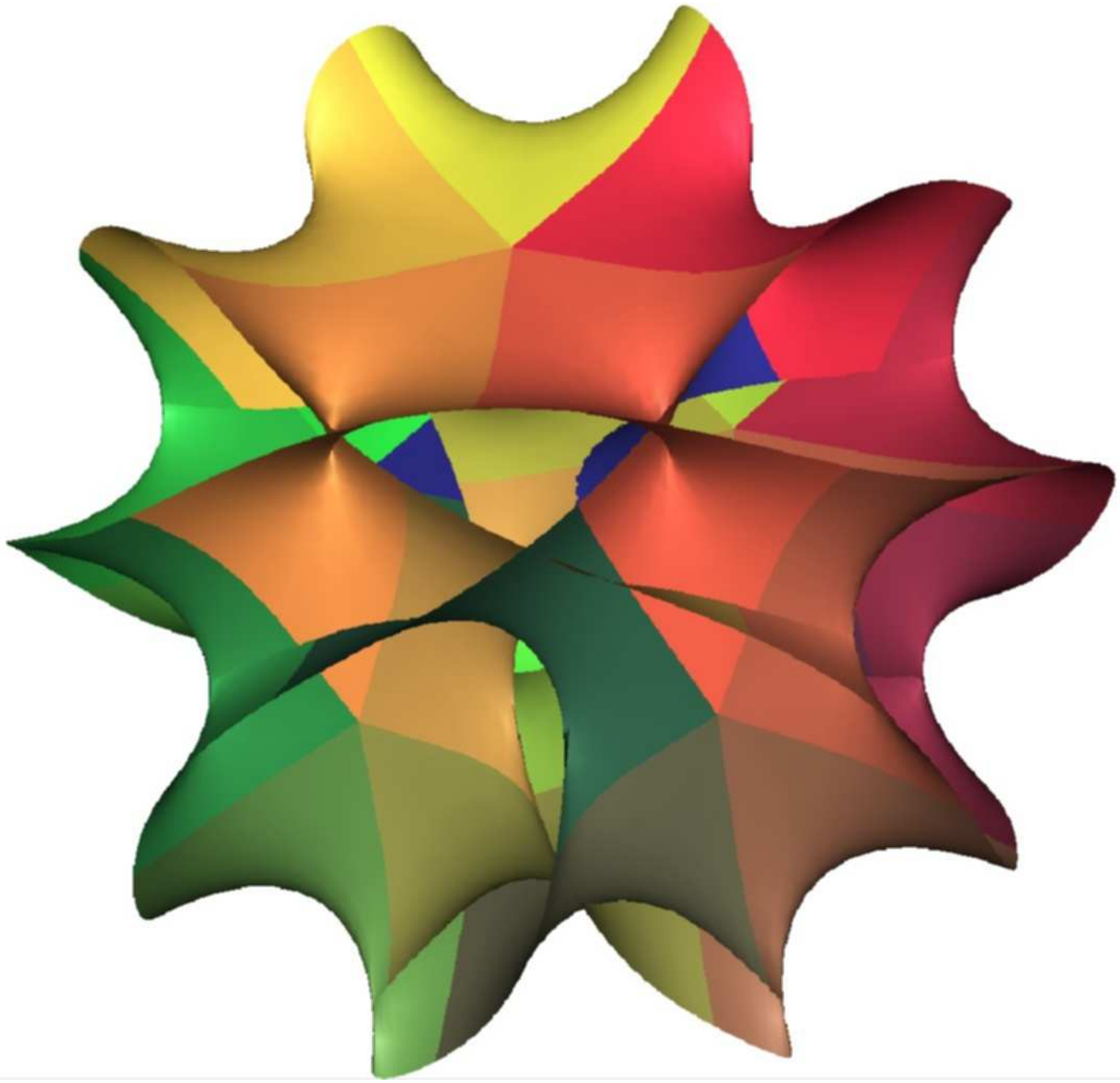


Crystal Calabi-Yau Sculpture

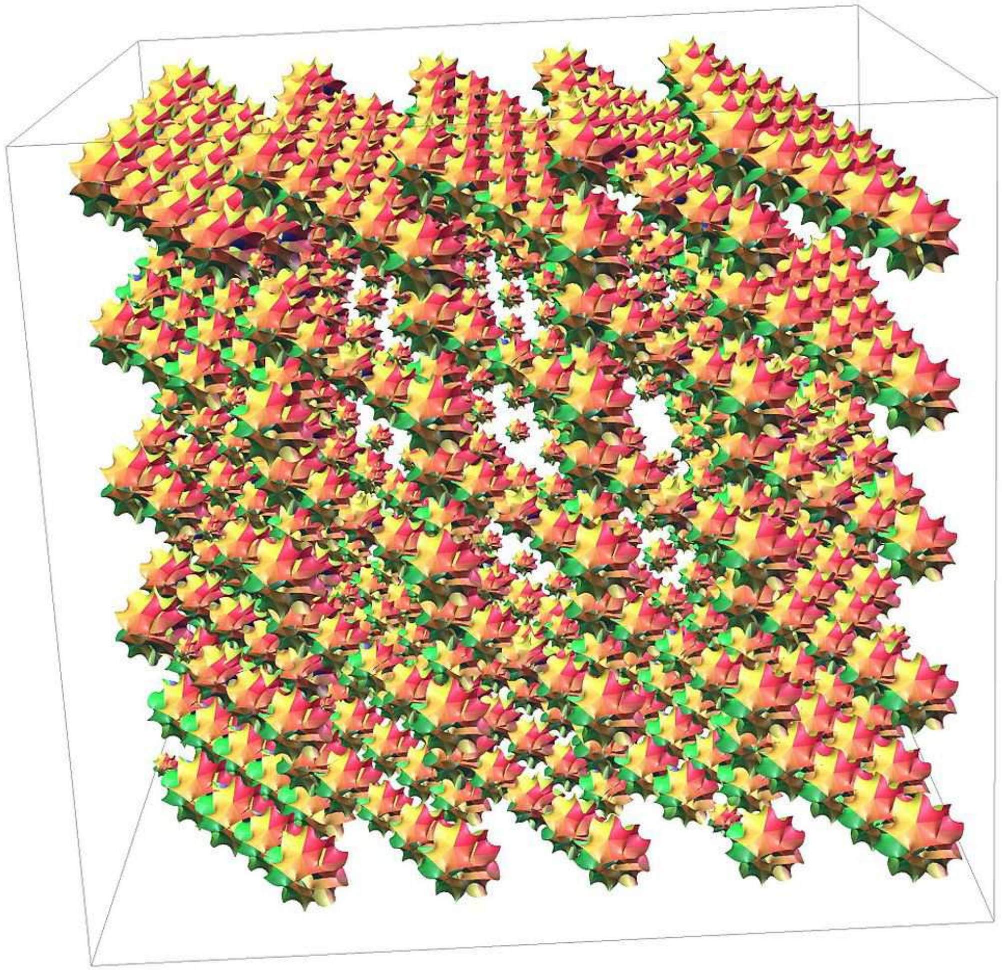


Artist: http://www.bathsheba.com

My version of 2D Cross-Section exposes many structural details...



The Big Picture: The 6D Calabi-Yau Quintic Structure



This is actually SIX dimensional: the partial space is sampled on a 4D grid, and the remaining 2D cross-sections are shown as they change across the grid.

Mathematical Details

- **How does one actually compute the equation of a Calabi-Yau space using the Equation of a CIRCLE?**

Roots: an uninformative approach to CY spaces?

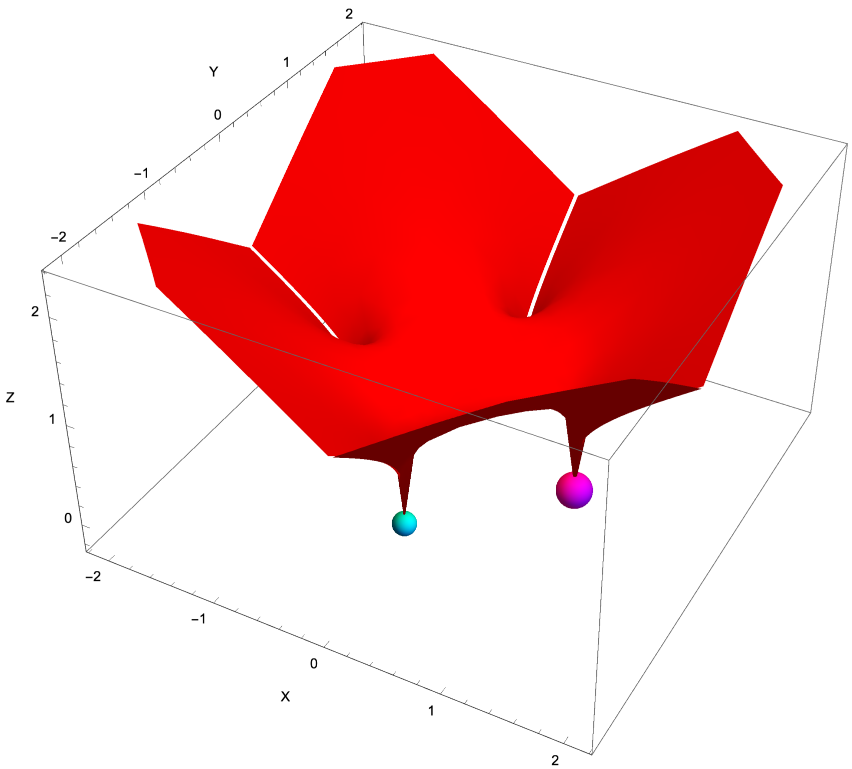
Inhomogeneous Eqns in $\mathbb{C}\mathrm{P}(N)$: look at homogeneous polynomial order p subspaces, divided by ${z_0}^n$ to give an inhomogeneous embedding in local coordinates:

$$\sum_{i=1}^{N} (z_i)^p = 1$$

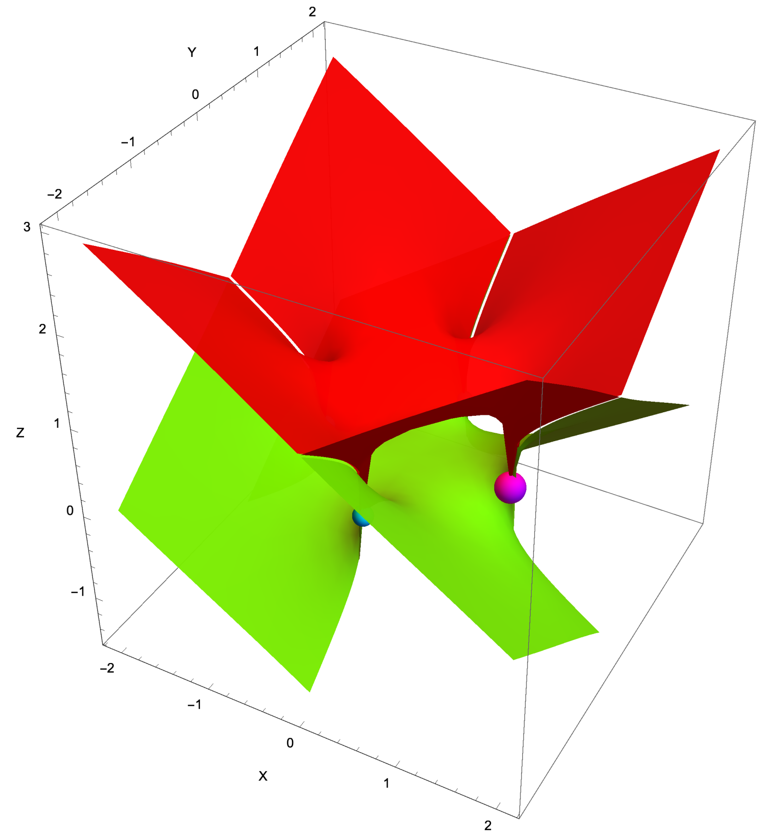
Suppose we try to draw this using p layers of polynomial roots, which for $\mathbb{C}\mathrm{P}(2)$ would look something like

$$\boxed{w(z) = \sqrt[p]{1 - z^p}}$$

Plotting layers of Riemann sheets . . .

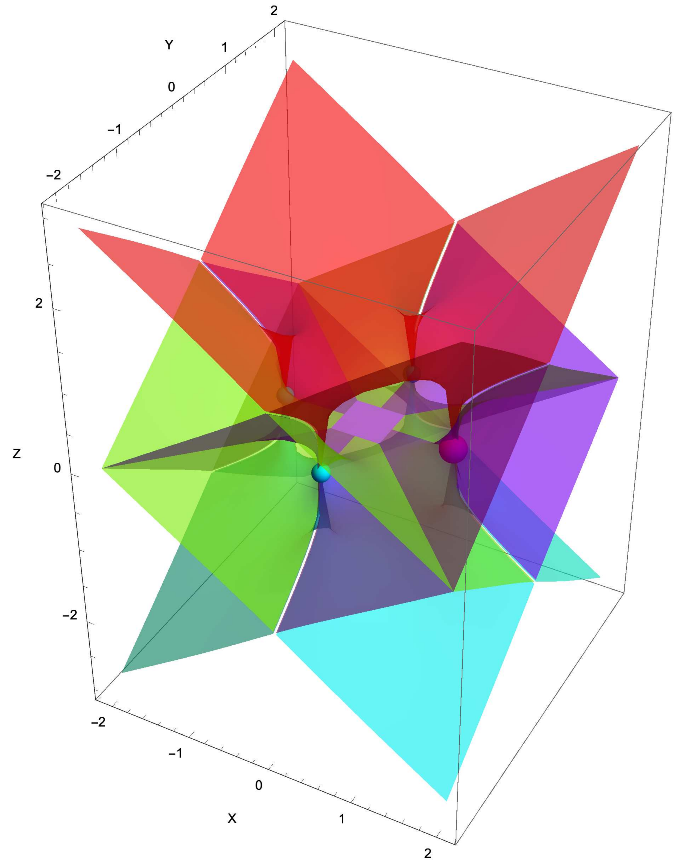


First root of $p = 4$ case.



First two roots.

Four-Root Riemann surface of Quartic:



This is "correct," but where is the geometry? Where is the topology?

[Riemann Surface Demo]

Better Visual Methods for CY spaces

- **Solve the $\mathbb{C}\mathrm{P}(2)$ slice equations with power p by exploiting fundamental domains:**

$$z_1{}^p + z_2{}^p = 1$$

can be split into p^2 pieces using method of AJH, *Notices of the Amer. Math. Soc.*, 1156–1163, **41**, 1994. Keep In Mind that we have taken $\boxed{z_0 = 1}$ here: the rest of the manifold lives at $\boxed{z_0 = 0}$!

- This is effectively stolen from computer graphics tricks in Barr's 1981 superquadric paper, *complexified*.

Algebraic Methods, contd.

Basic idea in the *Notices* article:

- **The Superquadric Trick:** First write down a circle:

$$x^2 + y^2 = 1.$$

Then parameterize with $x = \cos\theta,\ y = \sin\theta$, and take $\boxed{z_1 = x^{2/p} \quad z_2 = y^{2/p}}$ so that

$$\boxed{{z_1}^p + {z_2}^p = x^2 + y^2 = 1}$$

- **Then Complexify:** Let

$$\theta \to \theta + i\xi$$

Algebraic Methods, contd.

Then we can write, e.g.,

$$x = \cos(\theta + i\xi) = \cos\theta \cosh\xi - i \sin\theta \sinh\xi$$

to solve p**th order inhomogeneous Eqns** in $\mathbb{C}\mathbf{P}(2)$:

$$(z_1)^p + (z_2)^p = (x^{2/p})^p + (y^{2/p})^p = 1$$

which now reduce to **the equation of a *complex* circle!**

$$x(\theta, \xi)^2 + y(\theta, \xi)^2 = 1$$

. . . but the PHASE is tricky . . .

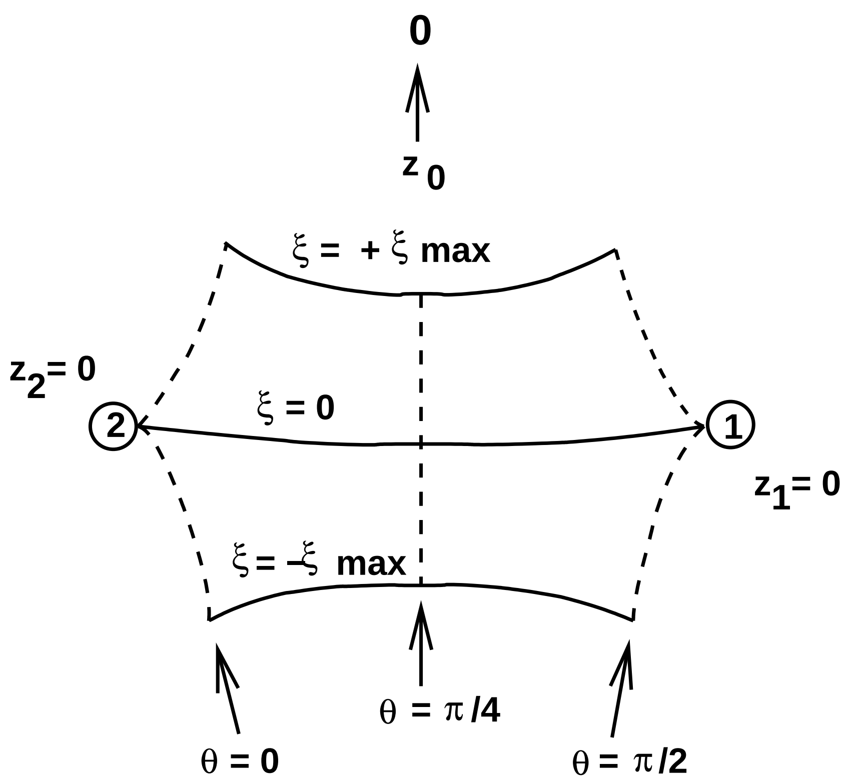
- **Fundamental Domain = First Quadrant:** The trick is that you only use $0 \leq \theta \leq \pi/2$.

- **Two sets of p separate phases solve eqns:** Now look at **whole set** of solutions: $k = 0, \ldots, (p-1)$:

$$z_1(k_1) = x^{2/p} e^{2\pi i k_1/p}, \quad z_2(k_2) = y^{2/p} e^{2\pi i k_2/p}$$

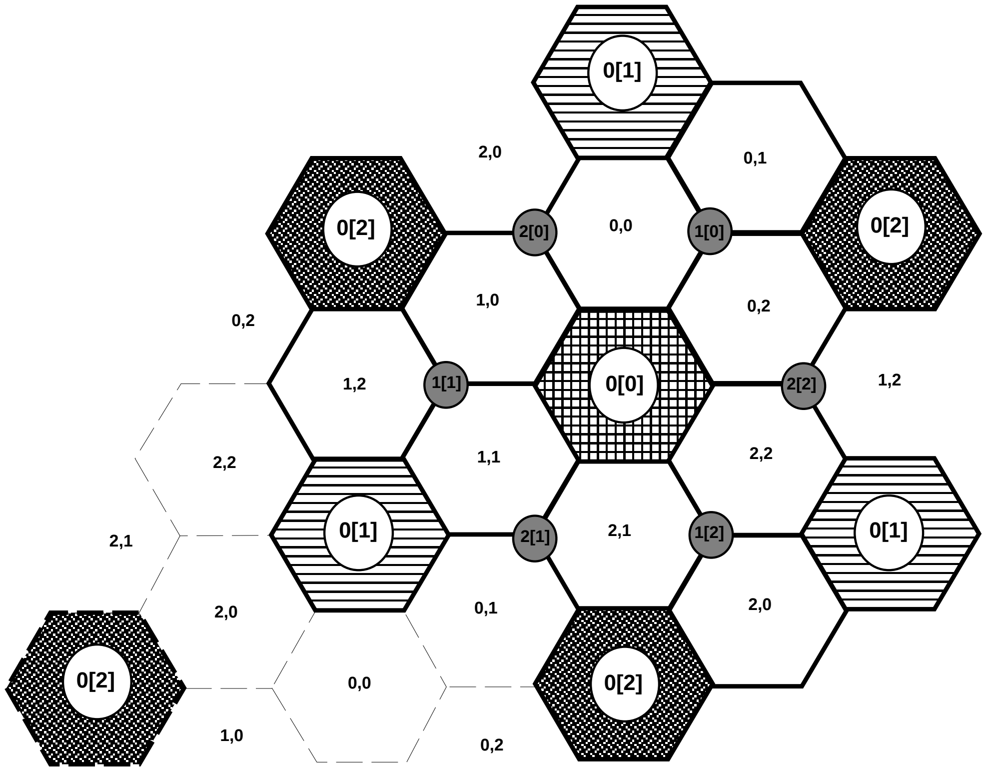
This gives $\boxed{p^2 \textbf{ patches } (k_1,\, k_2)}$ that **fit together.**

Algebraic Methods, contd



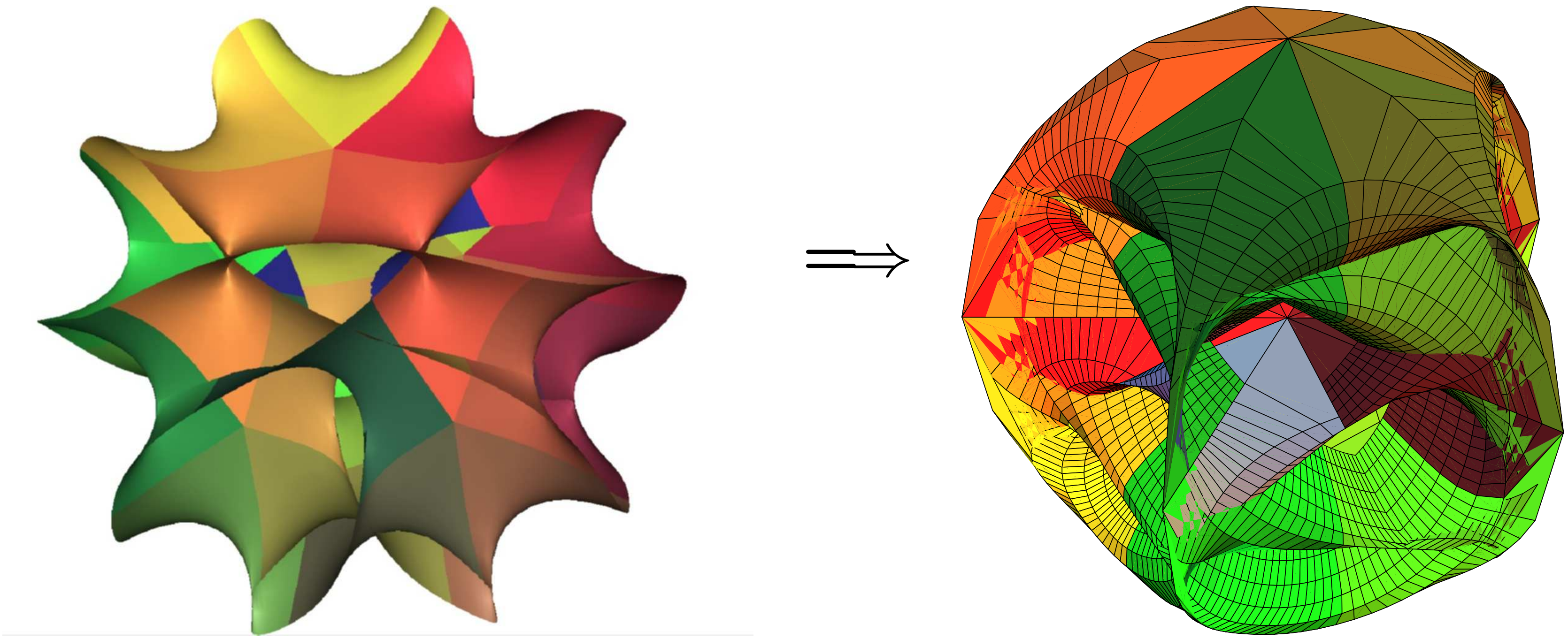
A single complex quadrant of the complexified Fermat equation comprises the fundamental domain.

Algebraic Methods, contd



$p = 3$ equation: $3 \times 3 = 9$ patches making a TORUS.

Compact Methods . . .



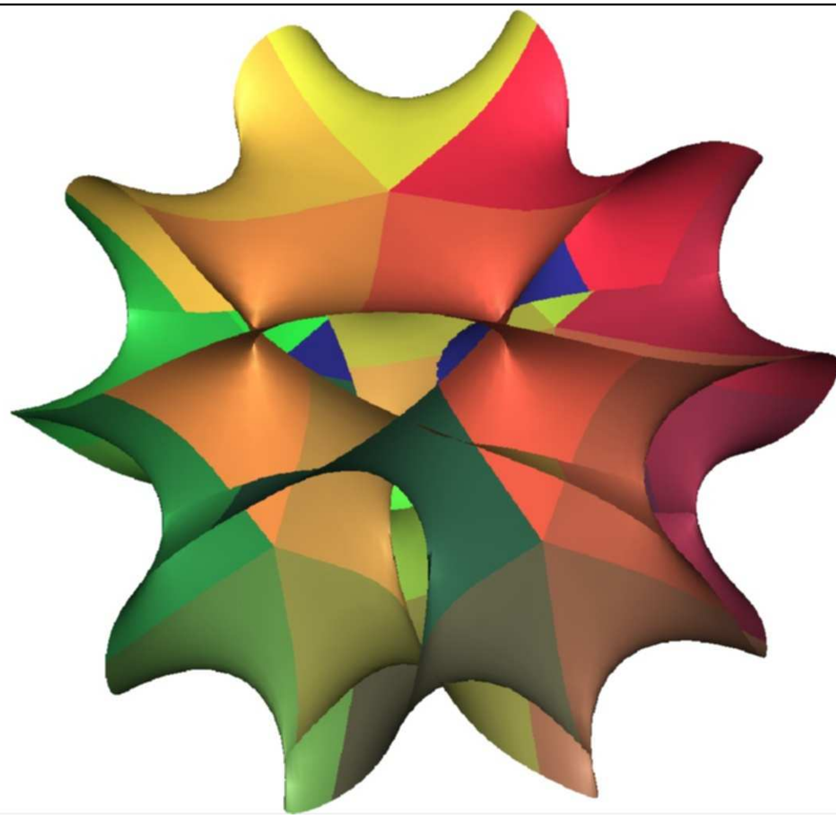
The actual compact genus 6 quintic cross-section projected to 3D looks like this!

SUMMARY of typical Calabi-Yau spaces.

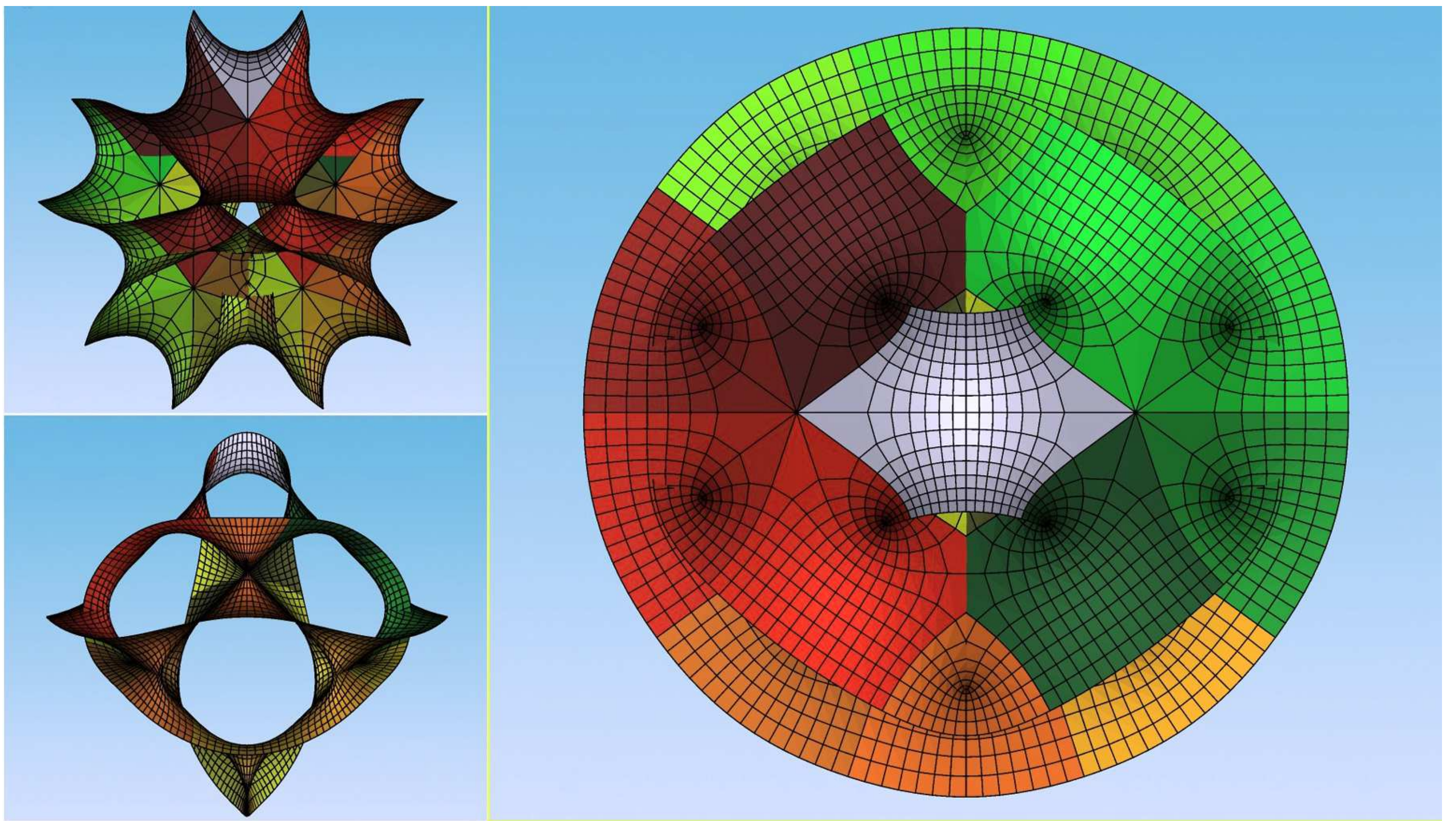
N	CP	deg(f)	$\mathbb{C}$ dim	$\mathbb{R}$ dim	*Remarks*
1	$\mathbb{C}\mathbf{P}(1)$	2	0	0	$z = \pm 1$, the 0-sphere $\mathbf{S}^0$
2	$\mathbb{C}\mathbf{P}(2)$	3	1	2	flat torus $\mathbf{T}^2$
3	$\mathbb{C}\mathbf{P}(3)$	4	2	4	K3 surface
4	$\mathbb{C}\mathbf{P}(4)$	5	3	6	Quintic $\rightarrow$ C-Y of String Theory?
N	$\mathbb{C}\mathbf{P}(N)$	N+1	N-1	2(N-1)	Solution of $\sum_{i=1}^{N} (z_i)^{N+1} = 1$

Calabi-Yau DEMO

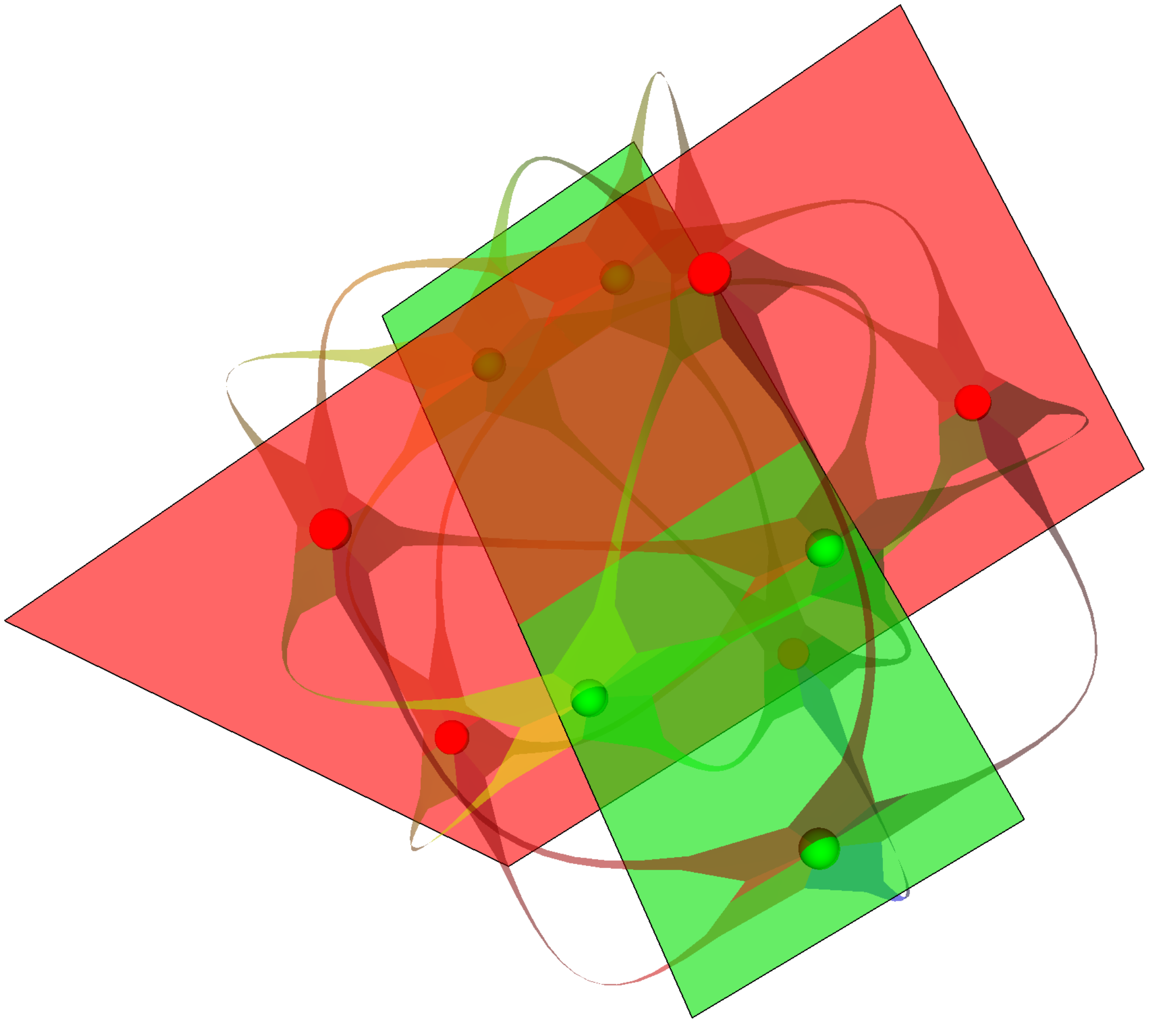
Visualizing $\mathbb{C}\mathrm{P}(2)$ Calabi-Yau Space Sections



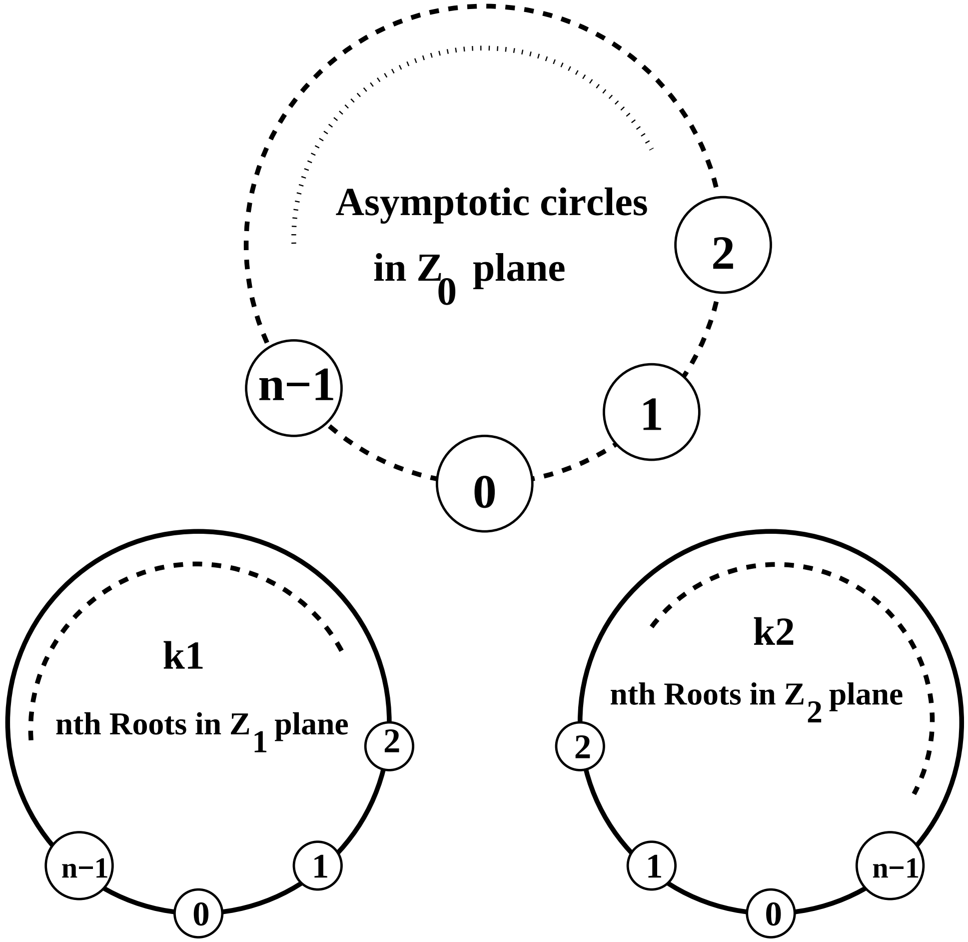
Now let's do some Topology…



Complex Roots at core of Calabi-Yau Quintic:



Topology! Count the vertices and edges of $z_1{}^n + z_2{}^n = 1$.



3 n Vertices

n^2 faces = pairs (k1,k2)

n^2 x 4 edges /2 = 2 n^2 edges

Prove Riemann-Hurwitz Formula...

The Complexified Fermat equation $\boxed{{z_1}^n + {z_2}^n = 1}$ has

Vertices	$3n$	*One set of n vertices for the roots on each complex line.*
Edges	$\frac{1}{2} \times 4n^2$	*Four edges per face divided by 2 .*
Faces	n^2	*One face (k_1, k_2) for each pair of roots $k_1 = \{0, \ldots, n-1\}$ and $k_2 = \{0, \ldots, n-1\}$.*

Prove Riemann-Hurwitz Formula...

Thus the genus of the surface $\boxed{{z_1}^n + {z_2}^n = 1}$ is the solution of:

$$\begin{aligned} \text{Euler No.} &= V - E + F \\ &= 3n - 2n^2 + n^2 = -(n-1)(n-2) + 2 \\ &= 2 - 2g \end{aligned}$$

solving:

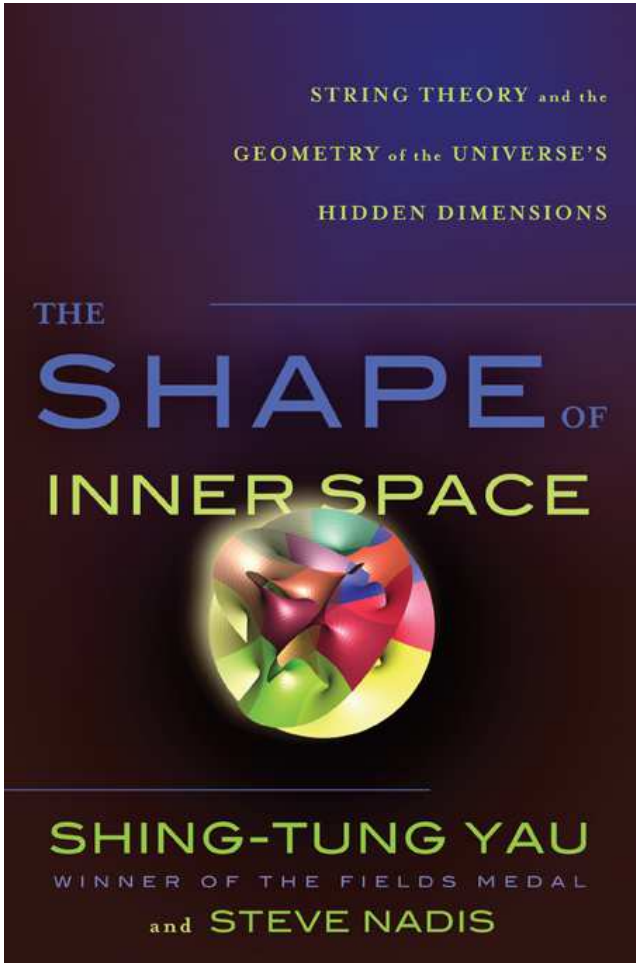
$$g = \frac{(n-1)(n-2)}{2}$$

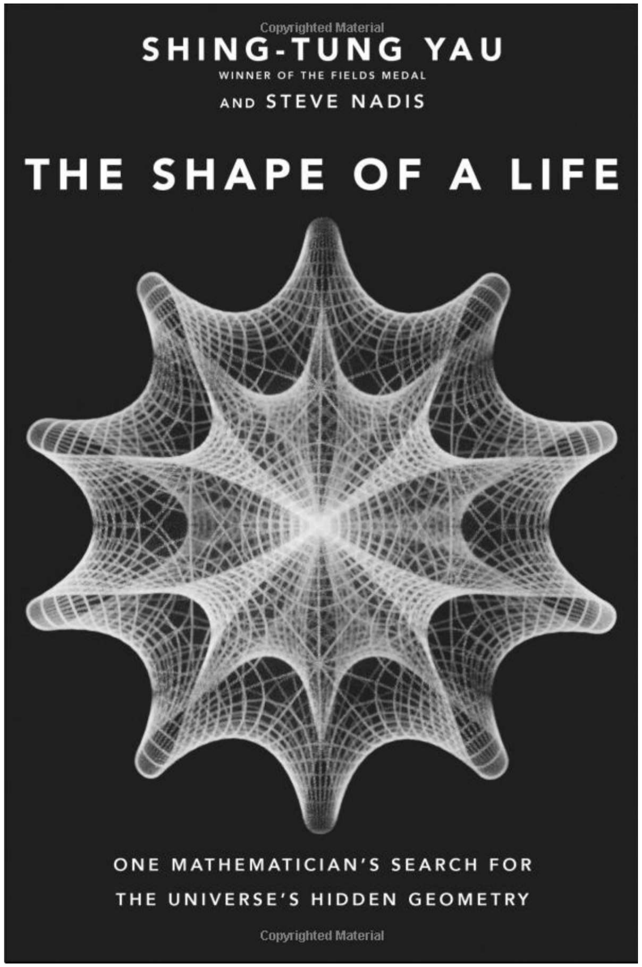
This is the famous Riemann-Hurwitz Genus Formula for homogeneous polynomial Riemann surfaces.

So that’s the story of the Calabi-Yau images!!

It’s been an interesting journey . . . here a few places they’ve been used:

Covers of Shing-Tung Yau's recent books.



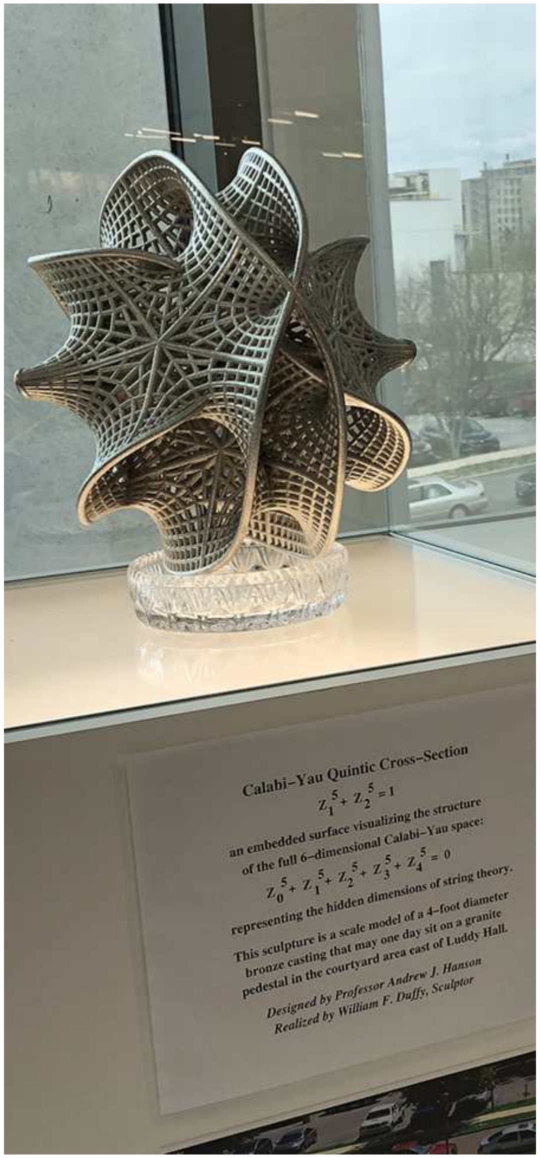


— Logo for the Harvard CMSA —

Clothing Advertising?

. . . on a London clothing ad billboard.

Sculptures!



Just Installed 3D Steel Print || Simulated Proposal for Courtyard

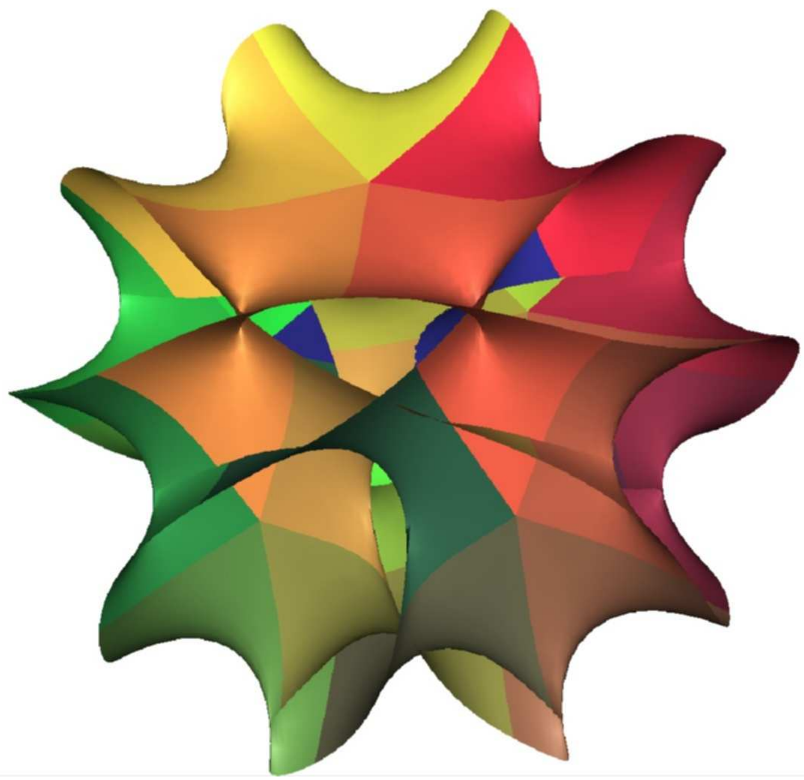
Conclusion of our Journey:

From Circles to SuperQuadrics,

from SuperQuadrics to Fermat Surfaces,

from Fermat Surfaces to Calabi-Yau Quintics.

Can we solve the Six Hidden Dimensions of String Theory?



maybe some day . . .

Thank you!

Try the Calabi-Yau demo for yourself . . .

Get my WebGL 4D Explorer link *here*.

`http://homes.sice.indiana.edu/hansona`

4D Explorer