CSCI 241H, Final Exam

Each question is 20 points, 10+10 for 2,3 and 7+13 for the rest. Try to answer all of them, keeping in mind that it's your placement with respect to the rest of the class that matters. Good luck!

Thank you for a fun semester, teaching this class was a pleasure.

1. Logic, sets.

   (a) A, B are two sets where $|A| = |B|$. Argue that $|A - B| = |B - A|$.

   Use the definition of $A - B$ as an intersection. Then $|A - B| = |A| - |A \cap B|$. Similar argument for $B - A$.

   (b) I have a collection of n people, half are men and half are women. You can assume that you know people's genders and name them accordingly; for instance, male i and female j could be named m_i, f_j. I would like to partition them into three groups such that each group has at least two men and two women. Write this as a satisfiability problem. That is, write down a set of logic formulas such that the problem is satisfiable if and only if there is a satisfying assignment of truth values to the variables. How many independent formulas do you have? How would you extract the group memberships from a satisfying assignment? If you are making any assumptions, make sure to put them down.

   We did not do satisfiability this term.

2. Counting.

   (a) I have \$10,000 that I would like to distribute among my n employees as a year-end bonus. Each employee will receive a bonus between \$0 and \$10,000 (note that, if one employee gets \$10,000, no other employee can receive a bonus); additionally, each bonus will be a multiple of \$100. In how many ways can I do this? Notice that I am capable of telling my employees apart; A getting X amount and B getting Y is different from A getting Y and B getting X. Note that I solved a questions which was very similar to this in class.

   Consider this as a problem of placing $n - 1$ markers on a line which is 10000cm long. Then the length of each segment is how much each employee gets. How many ways are there of doing this?

(b) I have k people of different ages between 0 through 100 years of age. I know that given k, there must be at least two pairs with identical age difference. For instance, I might have four people, 2, 12, 34, and 44 years old. The first two have an age difference of 10, as do the second two. The pairs can even share a person: people A, B, C of ages 8, 16, 24 make two pairs with an age difference of 8 – B is included in both pairs. What is the smallest possible value for k? Use the pigeonhole principle.

The age difference can take at most 101 values. Thus if we have 102 pairs, two of them must have the same age difference. In order to have 102 pairs of people we need to have $k = 15$ people at least (this would give us 105 pairs).

3. Probability.

(a) This is the question I asked you all in class. If I have a coin which comes up heads with probability p and tails with probability $1-p$, what is the expected number of flips until I get heads? Show your work.

We didn't do probability.

(b) You have 100 6-sided dice. 99 are fair, 1 is "loaded": it always gives you a 6. Now you pick a die at random, roll twice and get two 6s. What is the probability that you picked the loaded die? Note: this is actually a very simple question if you have the right approach.

Skip.

4. Asymptotic complexity.

(a) Show that $n^n \neq O(2^n)$.

Proof by contradiction, if we had c, k for this, the inequality would not hold for large n.

(b) I have $f(x) = x$ and $g(x) = 2^{2^x}$. Can you come up with a function $h(x)$ such that $h(g(x)) = O(h(f(x))$? h should actually mention x, i.e., while $h(x) = 5$ does work, you should come up with something more complex that that. And, no, $h(x) = x/x$ is not OK, either.

$h(x) = 1/x$.

5. Graph theory.

(a) One liner: Is a tree necessarily bipartite? Argue.

Yes. Even levels red, odd levels blue.

(b) Show by induction that any grid (mesh) of dimensions $m \times n$ where $m, n > 1$ and m is odd has a Hamiltonian cycle. Hint: Would you induct on n, m, or both?

We didn't do meshes.