# CSCI 241H:

## HOMEWORK

Solve the first four questions. Show your work.

1. $\overline{G}$, the complement of $G = (V, E)$ is a graph on vertex set $V$. It contains edge $(i, j)$ if and only if $G$ does not contain it in its edge set. For instance, the complement of a complete graph is just a bunch of nodes with no edges. Find a graph $G$ on more than 3 nodes where $G$ is isomorphic to its complement. Prove. Now, for a hard one: prove that the complement of a bipartite graph cannot be bipartite.

   Let $G$ have four nodes, $a, b, cd$. Its edges are $(a, b), (b, c), (c, d)$. The complement has edges $(b, d), (d, a), (a, c)$. Both are a line of three edges – you can easily come up with a mapping of the vertex labels. For the second one, consider any graph of more than 5 vertices (as discussed in class, 4 vertices will not work). One of the partitions will have at least 3 vertices, and, by definition, there are no edges between any two of these vertices. Thus, the complement has a complete graph on all those vertices, and thus will have a cycle of length 3.

2. Show that a graph $G$ with $n$ vertices is connected if it has more than $(n - 1)(n - 2)/2$ edges.

   Consider the highest number of edges a graph can have without being connected. It must have two connected components, and, to maximize the number of edges, they must be size $n - 1$ and 1. To maximize the edges, the large component must be a complete graph, which will have $(n - 1)(n - 2)/2$ edges.

3. How many vertices does a $d$-regular graph with $m$ edges have? (recall that a $d-$ regular graph is one where each vertex has degree $d$.) Assume that the graph is undirected.

   The number of edges is half the sum of the degrees. Thus, $2m = dn$ and $n = 2m/d$.

4. Show that in every undirected graph there is a path from every vertex of odd degree to some other vertex of odd degree. Hint: prove by contradiction.

   Assume not. Consider such a graph. Start from an odd vertex – you should not be able to go to any other odd vertex. Start making a path without repeating any edges. This path must end on an even vertex. But when we go into an even degree vertex, there is always an edge that we have not taken that takes us out of that vertex, as we showed for Euler's theorem. So we cannot be stuck at an even vertex.

5. 10.3, Questions 28, 52.

   28. It's the degree of the vertex corresponding to that row for undirected, outdegree for directed. 52. There are $m$ possible edges, $m = n(n-1)/2$, thus, both $G$ and its complement must have $m/2$. Since $m$ must be divisible by 2, $n(n-1)$ must be divisible by 4. This happens if either $n$ or $(n-1)$ is 0 mod 4.