CSCI 241H:

HOMEWORK 6

Show your work.

Prove the following by induction. Show all steps.

1. $\Sigma_{i=1}^{n} i^3 = (n(n+1)/2)^2$ for positive integer n.

   True for $k = 1$. Assume for k: $\Sigma_{i=1}^{k} i^3 = (k(k+1)/2)^2$.

   Check for $k+1$:

   $\Sigma_{i=1}^{k+1} i^3 = \Sigma_{i=1}^{k} i^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2(k^2+4k+4)}{4} = \frac{(k+1)^2(k+2)^2}{4}$

2. $\Sigma_{j=0}^{n}(-\frac{1}{2})^j = \frac{2^{n+1}+(-1)^n}{3 \cdot 2^n}$ for nonnegative integer n. **Hint:** You might want to consider two different cases for n.

   The trick is to look at even n and odd n separately. I will only show the analysis with even n. Odd is very similar, and of equal difficulty. Since I want n to be even, I will go from k to $k+2$. Thus, I need to check two base cases, say 0 and 1. Let's assume it holds for even k, it becomes a bit simpler:

   $\Sigma_{j=0}^{k}(-\frac{1}{2})^j = \frac{2^{k+1}+1}{3 \cdot 2^k}$

   Now to write it for $k+2$ – you could also write for $k+1$, you would still need to consider two cases.

   $\Sigma_{j=0}^{k+2}(-\frac{1}{2})^j = \frac{2^{k+1}+1}{3 \cdot 2^k} - 1/2^{k+1} + 1/2^{k+2} = \frac{4 \cdot (2^{k+1}+1)}{3 \cdot 2^{k+2}} - \frac{3}{3 \cdot 2^{k+2}} = \frac{2^{k+3}+4-3}{3 \cdot 2^{k+2}}$

   This completes the proof.

3. $3^n < n!$ if n is an integer greater than 6.

   $k = 6$ checks.

   Write for k: $3^k < k!$.

   Check for $k+1$: $3^{k+1} = 3 \cdot 3^k < 3 \cdot k! < (k+1) \cdot k! = (k+1)!$

4. $4^{n+1} + 5^{2n-1}$ is divisible by 21 if n is a positive integer.

   Checks for $k = 1$

   Assume for k: $4^{k+1} + 5^{2k-1} = 21p$ for some positive integer p.

   Write for $k+1$: $4^{k+2} + 5^{2k+1} = 4 \cdot 4^{k+1} + 25 \cdot 5^{2k-1} = 4(4^{k+1} + 5^{2k-1}) + 21 \cdot (5^{2k-1}$

   The first term is divisible by 21 by the I.H., the second because it's 21 times an integer, the proof follows.