Zeno's "Achilles and the Tortoise" Paradox and The Infinite Geometric Series

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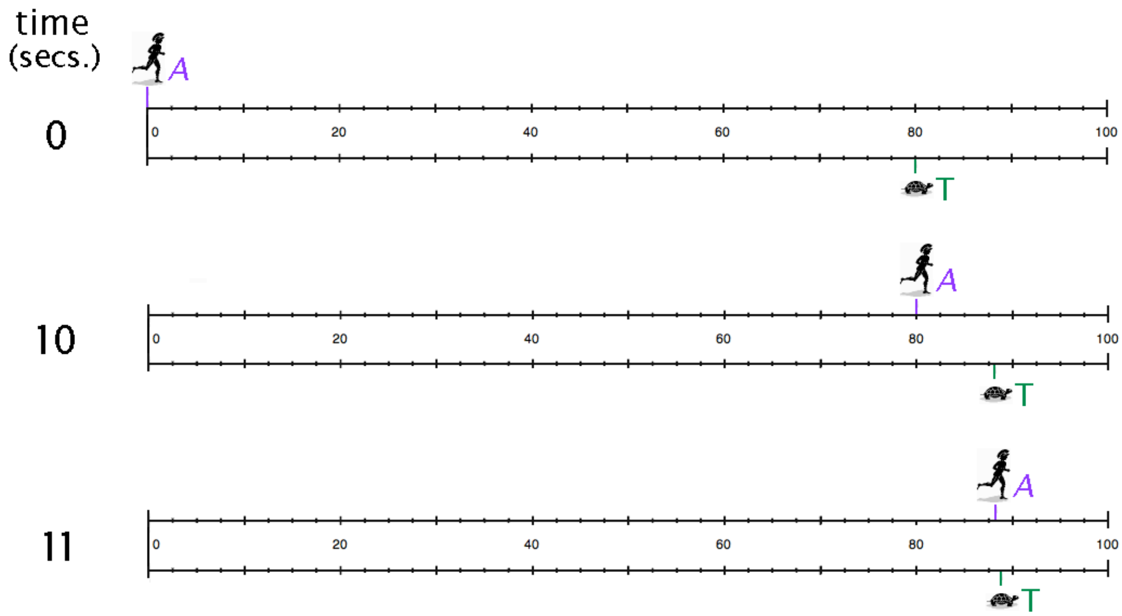
Note: On two occasions, I've presented Part I of the below to audiences of moderately technically-knowledgeable adults—mostly in-service or pre-service science teachers, but not math teachers—in about five minutes, with the help of a yardstick and a greenboard drawing. Each time, I think most of them "got it". I've also presented both parts to a high-school math "discovery" class at much greater length, and with reasonable success.

I. Resolving the Paradox

The Greek philosopher Zeno (5th century BCE) devised several "paradoxes of motion" that baffled all of his contemporaries, but with another 2400 years of mathematical progress behind us, we can do better! His first paradox involves a race between Achilles—a very fast runner, "the fleetest of foot of all mortals"—and a lowly tortoise. Naturally, the tortoise is allowed to have a head start. Zeno gives a simple but surprisingly plausible argument that Achilles could never catch up. As quoted by Aristotle (who gave his own refutation of Zeno's argument):

> "In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead."
>
> —Aristotle, *Physics* VI:9, 239b15

Let's be specific. Say the race is over a distance of 100 meters, and for simplicity, assume that each contestant moves at a constant speed. The tortoise goes 0.8 meters per second. Pretty fast—for a tortoise, that is; Achilles is 10 times as fast, i.e., he goes 8 meters per second! So we'd better give the tortoise a *huge* head start. Let's make it 80 meters, 4/5 of the total distance.



Now, after 10 sec., Achilles will have run 80 meters, bringing him to the point where the tortoise started. During this time, the tortoise has moved only 8 meters. It will take Achilles 1 sec. more to run that distance, by which time the tortoise will have crawled 0.8 meters farther. Then it'll take Achilles 0.1 sec. to reach this third point while the tortoise moves ahead by 0.08 meters. And so on and so on. Thus, whenever Achilles reaches somewhere the tortoise has been, he still has farther to go. Achilles must reach infinitely many points where the tortoise has already been before he catches up, so, as Aristotle said, he can never overtake the tortoise!

This is obviously wrong, but why?

The total time it would take Achilles to catch up, in seconds, is 10 + 1 + 0.1 + 0.01 + 0.001 +… . Technically, this is an *infinite series,* and yes, infinitely many numbers are being added up. But does that mean the total is infinite? Actually:

$$10 + 1 + 0.1 + 0.01 + 0.001 + \ldots = 11.111\ldots$$

That's not infinite! In fact, it's obviously between 11 and 12. But what is it exactly? We know 0.333333… = 1/3; dividing both sides by 3 gives 0.111111… = 1/9. So Achilles needs just 11 and 1/9 sec. to catch up.

The flaw in Zeno's argument is **his unstated assumption that the sum of an infinite series** (or at least an infinite series like this, where every term is greater than zero) **cannot be finite.** But the situation wasn't really clear until Newton and Liebniz invented calculus in the late 17^{th} century.

II. Boring Details, A Related Example, and the General Situation

In a more formal notation, the time for Achilles to catch up is:

$$\sum_{n=0}^{\infty} 10\left(\frac{1}{10}\right)^n = 10 + 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots = 11\frac{1}{9}$$

As a related situation that involves the same issues, consider this infinite series:

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

To visualize what this means, take an ordinary ruler (or yardstick) and draw a line two inches (or feet) long, then mark off one inch (or foot) of it. Then move half that distance and mark that point; then half of *that* distance and mark again; etc. No matter how long you do this—even if you could do it forever!—you'd never get past the two-inch mark. In mathematical terminology, 2 is an *upper bound* for the sum of this infinite series…and the sum of an infinite series in which each term is half the preceding one (sum from 0 to infinity of $(1/2)^n$) is finite.

In fact, this series (like the one for Achilles' and the tortoise's race) is *convergent*, i.e., it has a finite sum. With calculus and the concepts behind it, the argument that 2 is an upper bound for its sum can easily be made into a formal proof (by mathematical induction on the number of terms).

If each term of a series is the previous term times a constant ratio, as in both examples above, it's an infinite *geometric* series. Which infinite geometric series converge? All those (and only those) in which the ratio of consecutive terms is greater than –1 and less than +1, so that the absolute values of the terms get progressively smaller. It's that simple.