Tips and Resources for Solving Math Problems

Donald Byrd

IUPUI Math Education
Indiana University Informatics

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The material below is based both on my experience teaching high-school, middle-school, and introductory college math, and on my other experience as both teacher and learner. The "Tips" are probably most useful for students in upper-level high-school and lower-level college math courses; the "Resources" cover a much wider range.

Other than general problem-solving strategies, the Tips focus almost completely on algebra. Of course there are thousands of domain-specific identities, theorems, and "tricks"; but these are intended to be general-purpose ideas that can be applied almost anywhere. Thanks to Rochelle Effron, R. Patrick Morton, Paul Weedling, and Jacob Williams for many helpful comments, and to Lydia Sitnikov for the table of fraction operations.

Caveat. (1) I gave an earlier version of this document to my high-school precalculus students, but as far as I know, they completely ignored it. No wonder. If they even looked at it, it probably scared them off instantly; the average high-school junior or senior—not to mention younger students—needs something much shorter and simpler. (2) The Tips should be reorganized into "tactical" and "strategic" sections, and the more advanced tactics should be labeled as such or moved to a separate section. (3) It could benefit considerably by incorporating material from and references to G. Polya's classic *How to Solve It,* which I only recently became familiar with.

Resources on the Web

There's an enormous amount of valuable information for solving math problems on the World-Wide Web; I'll mention just a few sites. (Of course there's also an enormous amount in books and so on, but the average student is much more likely to go to the Web.) One caveat: Be aware that a webpage or video might not use the same terminology or standard forms you're used to.

Information

First, for reference or other general mathematical information, **Wikipedia** is usually excellent; its math articles are quite a bit better than many of its articles on other subjects. Be aware, however, that they're sometimes excessively technical for the ordinary mortal. **Wolfram Mathworld** articles are probably even higher quality, but (at least in my experience) its articles are even more technical as well as much less detailed; for both reasons, they're likely to be far less helpful to students at the levels I'm addressing. In addition, Wikipedia articles sometimes have nifty features like animations. For example, the "first derivative" and "second derivative" articles have animated diagrams showing a tangent line moving along a curve, changing color to indicate positive, zero, and negative slopes.

PurpleMath has a tremendous amount of information in the form of tutorials, worked problems, and practice materials for algebra and related areas of mathematics (including trigonometry and precalculus), from the most basic to beyond high-school level. "Word problems" (as they're called in K-12 schools) seem to be a big trouble spot for almost everyone through at least 200-level college courses, and PurpleMath offers numerous lessons on them; a sample is "Translating Word Problems", at http://www.purplemath.com/modules/translat.htm . Fractions are another popular trouble spot; PurpleMath offers http://www.purplemath.com/modules/fraction.htm . At the moment, its coverage of trigonometry is less complete, but it still appears to have enough to be very useful, e.g., a lengthy collection of identities. It has little or nothing on calculus.

The **Khan Academy** website (http://www.khanacademy.org/) has over 2,800 short videos; they cover a range of subjects, but most are on mathematical topics ranging from very basic (addition and subtraction) to at least advanced undergraduate level (differential equations, linear algebra, etc.). All that I've seen are very well done. As on so many websites these days, users can comment on videos and other users can respond, perhaps to answer questions. A relatively new feature of Khan Academy is exercises linked to many of the videos; I've looked at just one (for basic calculus), and it looked superb—plus there's a system of scoring, progressive exercises, etc. which should make the whole exercise business far more useful. (Note: many—maybe all?—of the Khan Academy videos are available through YouTube, but if you access them that way, you lose the links to exercises and user comments.)

Wolfram Alpha (http://www.wolframalpha.com) is a unique and very powerful resource: in its own words, "an engine for computing answers and getting knowledge." In algebra, for example, it can solve equations, factor polynomials, and manipulate rational functions and matrices; but it can also solve problems in trigonometry, geometry, calculus, number theory, etc., as well as more advanced parts of algebra like finite groups and linear algebra. Even better, **it can often show you *how* it solved the problem!** Versions are available for smart phones (iPhone and, I believe, Android) as well as the Web. But Wolfram also offers "course assistants" for a dozen subjects, including algebra, precalculus, and calculus; these are based on Wolfram Alpha, but specialized for their subjects, and are probably a better choice for most students.

S.O.S. MATHematics (http://www.sosmath.com/) is, in its own words, "your free resource for math review material from Algebra to Differential Equations". It has over 2500 pages in tutorial form, organized into the subject areas of Algebra, Trigonometry, Calculus, Differential Equations, Complex Variables, Matrix Algebra, or Mathematical Tables. A copy of the site is also available on CD for a moderate price.

"Common Errors In College Math", by Eric Schechter of Vanderbilt University, is excellent; at http://www.math.vanderbilt.edu/~schectex/commerrs/. Schechter's list has a much narrower focus than my collection of tips below, but it's far more extensive, and it's full of insightful comments about what the likely causes of many errors are, in what circumstances the subtler ones really *are* errors, etc. While it's addressed to undergraduate college math students, a great deal of what he says applies to high-school math as well.

The **"Art of Problem Solving"** website is all about solving mathematical problems; see its "AoPS Resources", at http://www.artofproblemsolving.com/Resources/index.php . AoPS has a particularly nice summary of factoring methods, at http://www.artofproblemsolving.com/Wiki/index.php/Factoring .

The Math Forum - Ask Dr. Math (http://mathforum.org/dr.math/) is primarily aimed at high school students and classroom teachers at all pre-college levels. It is now run out of Drexel University.

The **Finite Mathematics & Applied Calculus: Everything** website (http://www.zweigmedia.com/ThirdEdSite/tccombop.html) may not quite live up to its name, but it comes closer than you might guess. Its creators are the authors of a similarlytitled textbook. The website appears to have the entire contents of the book plus an enormous amount of additional information, including many tutorials, exercises, and generally-useful tools, especially for graphing: for example, general function, derivative, integral, linear programming, and Riemann sum graphers, plus matrix algebra and row operation/pivot utilities, probability generators, a Markov system simulator, etc. Most of the tools are online, but some are Excel spreadsheets you can download. There's also an **Applied Calculus: Everything** website (http://www.zweigmedia.com/ThirdEdSite/tccalcp.html), which contains just what you'd expect: a subset of what's on the other site, with the material that isn't directly relevant to calculus removed.

As you might guess, the **Calculus.org** website (http://www.calculus.org/) consists of "Resources For The Calculus Student", including problems with step-by-step solutions, animations and applets illustrating many ideas of calculus, tips on preparing for exams, sample exams, etc. However, it doesn't appear to be very active or well-maintained.

Some things in mathematics are best memorized: for example, at a very elementary level, addition and multiplication tables; at every level, important definitions. Song lyrics can be very helpful for memorizing math facts, and there are several sites for songs with such lyrics. One good one is **Larry Lesser's MATH & MUSIC (the Mathemusician!)** page (http://www.math.utep.edu/Faculty/lesser/Mathemusician.html).

Practice Material

I've already mentioned **PurpleMath, Khan Academy,** and **[Finite Mathematics &] Applied Calculus: Everything.** Here are some other possibilities.

NCTM Illuminations (http://illuminations.nctm.org), sponsored by the National Council of Teachers of Mathematics, has numerous activities for math students, organized by grade level and topic, and ranging up to calculus. NCTM's official interests stop at the end of high school, but some of their stuff would certainly be useful for 100-level and perhaps 200-level college courses as well. Their "Calculus Tool" in particular looks extremely useful. NCTM also has other neat things like "Dynamic Paper": a facility for creating many kinds of graph paper, grids of various shapes, pictures of geometric figures, etc.

NLVM, the "National Laboratory of Virtual Manipulatives" (http://nlvm.usu.edu/), contains hundreds of applets (mini computer programs/games) for teaching K-12 math, organized similarly to NCTM's. NB: While the applets require Web access, you can buy a version that doesn't (and a 60-day demo version is free); it doesn't appear to have any additional features.

Other websites that are probably worth looking at include:

AAA Math: http://www.aaaknow.com

IXL: http://www.ixl.com/math/

Math Playground: http://www.mathplayground.com

Online Math Learning: http://www.onlinemathlearning.com/

One Mathematical Cat: www.onemathematicalcat.org

Tips

Avoid Careless Errors.

Be Systematic, Clear, Neat. Sloppiness— sloppy writing so that e.g. “5” and “s” look similar, writing exponents that are barely above the baseline, inaccurate alignment of elements of a matrix, etc.—quickly leads to silly mistakes, and makes it a lot less likely that you (or a friend looking over your work) will spot the problem.

Take your time. For example, think twice before merging steps mentally to save the time of writing them down; even if you know exactly what you’re doing, it’s easy to get confused and write it down wrong.

A Word About Fractions

Problems doing arithmetic with fractions are frighteningly common. Many seem to be caused by confusion between how various operations work. In most ways, adding and subtracting behave similarly to each other, and multiplying and dividing behave similarly to each other. This makes sense, since adding and subtracting are inverse operations, and multiplying and dividing are inverse operations. For example, you can multiply or divide one fraction by another without worrying about a common denominator; but for adding or subtracting, you’d better worry. Here’s a table (“LCM” = Least Common Multiple):

	Adding/Subtracting	*Multiplying/Dividing*
Helpful to convert to a common denominator (the LCM) first?	Yes	no
Can cross-reduce?	no	Multiplication only
Helpful to convert mixed numbers to improper fractions first?	no	Yes

Another important case is confusion between multiplying only the numerator of a fraction by a number (to change its value by that factor) and multiplying both numerator and denominator by the same number (presumably to change its denominator to the LCM *without* changing the fraction’s value). For example, I asked my students to solve by elimination this system of equations:

$$\tfrac{1}{2}t - \tfrac{1}{5}v = \tfrac{3}{2} \quad \text{and} \quad \tfrac{2}{3}t + \tfrac{1}{4}v = \tfrac{5}{12}$$

Several students indicated in their solution that they intended to multiply the first equation by 3 and the second by 2. Those factors make no sense for eliminating either t or v; they don’t make sense for finding the LCM either, but multiplying *both numerator and denominator* of the first equation by 3 and of the second by 2 do! Sure enough, they got:

$$\tfrac{3}{6}t - \tfrac{3}{15}v = \tfrac{9}{6} \quad \text{and} \quad \tfrac{4}{6}t + \tfrac{2}{8}v = \tfrac{10}{24}$$

So they actually multiplied the first equation by 3/3 (=1), not by 3, and the second by 2/2 (=1), not by 2.

See below for more about fractions.

Popular Mistakes

Some of the silliest possible mistakes are also among the easiest to make, and the most common:

- **Losing track of the sign.** Almost certainly the most common of all algebra mistakes.

- **Putting the decimal point in the wrong place.**
- **Assuming the wrong order of operations.** The standard order ("priority") of operations: Parentheses, Exponents, Multiplication/Division, Addition/Subtraction. Note that multiplication and division are at the *same* priority; therefore whichever occurs first in an expression is done first. 12 / 3 * 2 = 8, not 2. The same holds for addition and subtraction. To remember this, think of "PE(MD)(AS)", not just "PEMDAS".
- **Parentheses.** Of course, parentheses override the priority order of operations. To head off problems, it's often a good idea to add parentheses where they might not be strictly necessary. (Experienced computer programmers often do just this in programming statements involving complicated mathematical expressions.)
- **Dividing by zero.** Particularly insidious, since it can easily be hidden: most of the various "proofs" that 1 = 0 or 2 = 1 or some such rely on a concealed division by zero.
- **Applying an operation to only part of an equation.** Let's say you have the equation $x^2 + 5x = 12$. If you add/subtract, multiply/divide, square, take the logarithm, etc., on the left, you must do the same on the right. And both terms on the left must be affected; this doesn't matter for addition/subtraction (it's not hard to see why), but it does for anything else.
- **Misapplying the distributive law.** The distributive law doesn't let you replace the expression $\frac{2}{5 + 3x}$ with $\frac{2}{5} + \frac{2}{3x}$. To see clearly that the two forms aren't equivalent, notice that if $x = 0$, the first expression evaluates to 2/5; the second is impossible to evaluate, since it requires division by zero.
- **Copying the problem incorrectly, or copying incorrectly from one step of your work to the next.** There's not much to say about this except that I've done it many times myself.
- **Confusing power functions and exponentials.** This isn't likely to cause much trouble in, say, algebra or trigonometry courses. But in calculus, trying to take the derivative of $3e^{2x}$ via the Power Rule is a big problem! In most cases, it's easy to tell the difference: if the variable is in the exponent, as in $3e^{2x}$, it's an exponential. If it's in the base—the thing being raised to some power—as in $3x^{2e}$, it's a power function. Of course you still need to be careful about what the variable is. (And the variable might appear as both base and exponent, e.g., x^{2x}. Or there might be two variables, one in the base and another in the exponent. But those situations are unlikely in anything up to 3rd-semester college calculus.)

Write Everything Down.

Write down everything you can think of that seems relevant, even if you don't think you need to. Seeing something in black and white in front of you, even if you know it very well, might remind you of something else that'll greatly simplify the problem. In calculus, for example, looking at the delta-epsilon definition of a limit might remind you that for *f(x)* to have a limit at *a* doesn't even require *f(a)* to be defined, and that might be critical for your problem.

Develop Your Intuition.

Worry about *understanding* the problem before you worry about *solving* it; develop your intuition about the situation. Some specific ideas:

- If your problem involves messy numbers, try to solve it first with small round numbers.

- Look for patterns. For an example, see the problem from Swokowski & Cole under "Factor the Difference of Two *n*th Powers" below.
- For some types of problems—e.g., what's the formula producing a mysterious sequence, or sketching the graph of a complicated function—gather data generated by the problem; then look for patterns. In the words of Erickson & Flowers (*Principles of Mathematical Problem Solving):* "When solving problems, it is often best to begin by obtaining data. Gathering data can be as important in mathematical investigation as in a scientific investigation, and sometimes just as necessary in getting started." Arrange the data carefully in a table or other suitably organized manner. For example, if you're looking at values of x and $f(x)$, your pairs of numbers should probably be arranged in order of increasing value of x, though increasing absolute value of x or increasing value of $f(x)$ might be useful instead. When you've done this, a pattern may jump out at you; if not, try drawing a graph or diagram of some kind.
- If you've been focusing on details, step back, or "zoom out", so to speak, and look at the whole problem instead. If you've been looking at the problem as a whole, focus on some part of it—even what seems like a mere detail; see "Work on Any Part of the Problem", below.

Work on Any Part of the Problem.

If you can't make headway against what seems like the main problem, making progress on some aspect of it, even a part that seems very minor, can put things in a different perspective and make everything much more tractable.

Techniques

The set of techniques below is far from complete, but I think it includes most of the specific methods I've found useful. In the spirit of working "from the inside out", we'll start with small elements and work our way up to larger ones: specifically, from *numbers* to *fractions* to *expressions* to complete *equations*.

Two Unique and Uniquely Flexible Numbers: 1 and 0

Many useful techniques for solving problems depend on special properties of the numbers 1 and 0, properties that almost everyone knows but whose implications people often forget. Foremost for 1 is the fact that you can always multiply by 1 without changing anything. Of course that applies to 1 itself: for any integer n, $1^n = 1$. In algebra, for any nonzero integer (or nonzero expression) q, $q/q = 1$. And in trigonometry, by the definitions of the functions:

$$(\sin\theta)(\csc\theta) = (\cos\theta)(\sec\theta) = (\tan\theta)(\cot\theta) = 1$$

Zero has the analogous property for addition, i.e., you can add zero to anything without changing it. But even more useful is the fact that the product of two numbers is zero if and only if at least one of them is zero.

The techniques below include examples of both of these unique numbers in action.

1 as the default coefficient. In the words of www.onemathematicalcat.org/algebra_book/online_problems/id_var_part_coeff.htm : "If you don't see a coefficient, then it is 1… It's never necessary to write a coefficient of 1, because multiplication by 1 doesn't change anything… In the term $-x$, the coefficient is 1, because $-x = (-1)x$." As a more complex example, in the exponential-based expression $\frac{2^x}{13}$, what is the coefficient? It can't involve the 2: you can't separate 2 from its exponent. To find the coefficient, just multiply the numerator by 1:

$$\frac{2^x}{13} = \frac{1 \bullet 2^x}{13} = \frac{1}{13} \bullet 2^x$$

The answer is 1/13.

Transform *fractions:* by multiplying by 1.

You can often simplify fractions by multiplying them by 1, with 1 represented as a convenient fraction q/q (where $q \neq 0$, of course). Rationalizing a denominator that contains an irrational number is a good example (here, $q = \sqrt{5}$):

$$\frac{x}{\sqrt{5}} = \frac{x}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{x \cdot \sqrt{5}}{\sqrt{5}\sqrt{5}} = \frac{x\sqrt{5}}{5}$$

Another example is simplifying a denominator that contains a complex number, where q is the conjugate of the complex number (in this case, $a - bi$):

$$\frac{u}{a + bi} = \frac{u}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{u \cdot (a - bi)}{(a + bi)(a - bi)} = \frac{ua - ubi}{a^2 + b^2}$$

Forms of Expressions

Symbolic vs. numeric and fractions vs. decimals. Keep things in symbolic form, i.e., put off converting expressions to numbers, for as long as possible. This applies even if what you want to end up with is a number, and even if you already know it'll be an irrational number. And if you have a number in the form of a fraction, don't convert it to a decimal unless and until you have to. Why?

Of course converting a fraction to a decimal equivalent tends to lose accuracy, but what's equally important is it tends to lose *information*. For example, the answer to one problem on a precalculus test I once gave was 55/31, or about 1.774194. One student gave the answer as 55/23; from the numerator, I knew immediately that person had a pretty good of idea how to solve the problem and I could give them credit for that, even though they had the wrong answer. But if they'd used the decimal equivalent of 55/23, 2.391304, it wouldn't have been clear at all.

Additionally, if you convert fractions to decimals that aren't exactly the same, in many situations it'll be impossible to check your answers. If an expression is supposed to equal zero but—when you fill in your results—it comes out –0.03, who knows what that means?

Finally, you might make a mistake! I've graded tests in which students wrote down a correct and perfectly acceptable symbolic expression, and then turned it into an incorrect number.

It's especially important to avoid converting a fraction to decimal form *early* in the solution to a problem, since the more steps follow the conversion, the greater the chance of problems resulting from either the loss of accuracy or the loss of information, and the harder it'll be for anyone (including you, a day or two later) to follow your calculations. (But obviously you'll have to convert the solution to a problem to a number if it asks for an answer to so many decimal places.)

Converting symbolic to numeric form always loses information. Similarly, 2.82843 is very close to $2\sqrt{2}$. But, unless the context makes it clear or I happen to notice that it's very close to $2\sqrt{2}$, the number doesn't give a clue as to how someone got it; the expression does.

Again, it's especially important to be careful about converting symbolic to numeric form early in solving a problem, since the "structure" of the symbolic form may have information that'll be very useful later. But this is more of a judgment call: it may be helpful or even essential to convert as you work.

Expressions involving mathematical functions (trigonometric functions, logarithm, etc.): for clarity, minimize the number of function references involved. For example, write

$$\frac{\log(9/4)}{\log 432}$$

instead of the equivalent but less meaningful

$$\frac{2\log 3 - \log 4}{2\log 4 + 3\log 3}$$

Work "from the inside out". This isn't usually a big deal, but with complicated expressions, it's the safest way to work, and it'll minimize round-off error, which can be serious. PurpleMath has an explanation of this phenomenon I can't improve on (http://www.purplemath.com/modules/expoprob2.htm).

Transform *expressions:* by rewriting them in terms of new variables, or of different functions.

For example, solving $3x^4 + 5x^2 - 2 = 0$ may look very difficult, but defining $u = x^2$ and substituting u for x^2 turns it into $3u^2 + 5u - 2 = 0$: a garden-variety quadratic. Solve that for u, then take $x = \pm\sqrt{u}$. Don't forget the plus/minus: the fourth-degree polynomial really does have up to four roots, potentially one or two values of x for each of the one or two values of u. In this case, $3u^2 + 5u - 2$ factors as $(3u - 1)(u + 2)$. This gives $u = 1/3$, $u = -2$, and therefore $x = \pm\sqrt{1/3}$; assuming complex numbers aren't allowed, of course $x = \pm\sqrt{-2}$ doesn't yield any solutions.

A related idea is to rewrite to get rid of less familiar and less convenient functions. This applies particularly to trigonometry, where, for example, you can use the fundamental identities to replace sec and csc with their equivalents in terms of sine and cosine. You may also want to replace cot and tan with their sine and cosine equivalents. Examples: replace $\sec(x)$ with $1/\cos(x)$; replace $\tan(x)$ with $\sin(x)/\cos(x)$.

Transform *expressions:* by factoring.

Factoring complicated expressions is important for several reasons. If the expression involves division, of course there might be a common factor in numerator and denominator, a factor that can be removed from both. It can be especially helpful for an expression that's known to equal zero, since one or more of its factors must be zero. A particularly useful case is finding linear factors of a polynomial, since each such factor immediately gives a zero of the unknown. In any case, if you can factor an expression, that in itself might be a clue to what to do with it.

Here are a few ideas for factoring. For more ideas, see, for example, AoPS (http://www.artofproblemsolving.com/Wiki/index.php/Factoring).

Factor the difference of two *n*th powers, and/or divide by the difference of two expressions.

For any integer $n>0$ and any expressions a and b, $a^n - b^n$ is divisible by $a - b$. This may be obvious, but it may not be obvious that something in your problem can be represented as $a^n - b^n$. For example, this problem occurs in a well-known precalculus text (it's problem 17 in Sec. 5.3 of Swokowski and Cole's *Algebra and Trigonometry, with Analytic Geometry*, 11th edition): Simplify

$$\frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

Many students fail to notice that the numerator is the difference of two squares, and therefore can be factored. Doing that is very helpful. But after factoring, you may be tempted to divide by $e^x + e^{-x}$; in this case, that's a mistake, since the numerator by itself reduces all the way to the constant 4.

(This is clearly related to the formula for the sum of a geometric series.)

Another tip: when factoring the difference of *n*th powers, it's a good habit to write the "plus" factor first, then the "minus" one. This is because it's more likely the minus term can be factored further. For example,

$$16r^4 - 81s^4 = (4r^2 + 9s^2)(4r^2 - s^2) = (4r^2 + 9s^2)(2r + 3s)(2r - 3s)$$

Factor the sum of two cubes (and other odd powers).

For odd integers $n>0$ and any expressions a and b, $a^n + b^n$ is divisible by $a + b$. In particular, $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$. Note that for *even* integers n, $a^n + b^n$ is generally not factorable (though there may be special cases where it is): specifically, $a^2 + b^2$ can't be factored.

Interpret 1 as 1^n.

For example, given that n is a positive integer, $1 - s^n = 1^n - s^n$; this is divisible by $1 - s$ since (as described above) $a^n - b^n$ is always divisible by $a - b$.

Transform *expressions:* by negating terms twice.

For example, synthetic division works only for divisors of the form $x - c$. To divide by, say, $r + 3$, just write it as $r - (-3)$.

(Advanced) Transform *expressions:* by adding zero.

Familiar methods like completing the square can be considered a form of adding zero to an expression: you add a nonzero value in one place but subtract the same amount in another place. But adding zero can be used in other ways! Let u be an expression you want to manipulate—perhaps an ugly, intractable one, perhaps not—and h be an expression that, when added to u, produces something you can work with. Taking advantage of associativity:

$$u = u + 0 = u + (h - h) = (u + h) - h$$

Transform *equations:* by taking logarithms.

If you have an equation with an unknown, say t, in an exponent, you might be able to solve it be just taking the tth root of each side, i.e. raising each side to the $1/t$ power; if that doesn't work, you'll almost certainly have to take the logarithms of both sides. If it's just a messy exponent, even without unknowns, taking the logarithms of both sides might still be a step towards solving it. If you do use logarithms, consider what base you should use; there may be a much better choice than either of the obvious choices, e and 10. Finally, note that to finish your problem, you might have to convert back to exponents!