Polynomials and their Derivatives:
Polynomials, Critical Points, and Inflection Points

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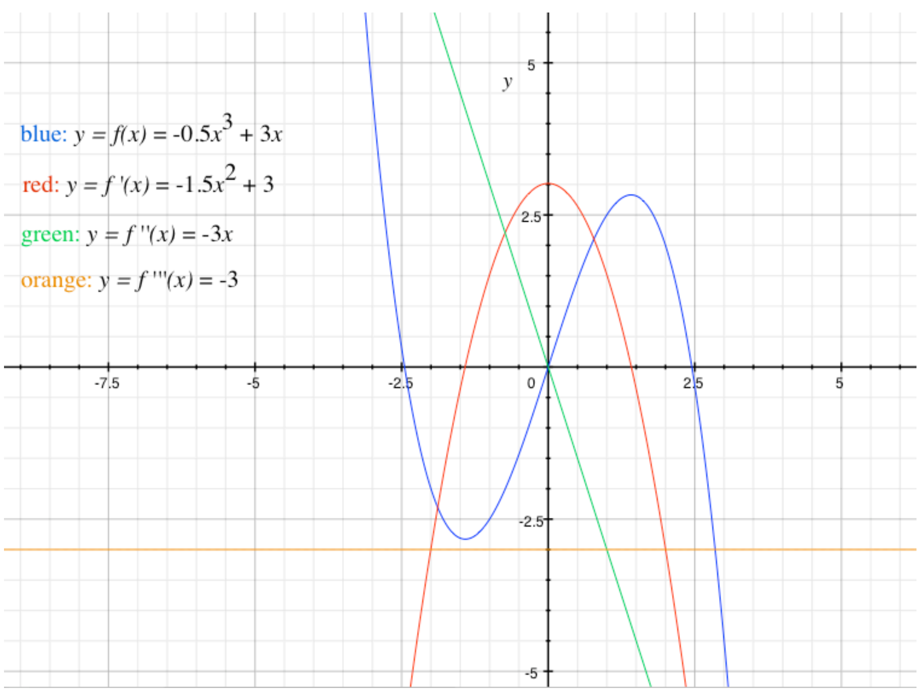
Here's some information about polynomials and their *derivatives* that should make it easier to understand both derivatives, and the relationships between the polynomials and their *critical points* and *inflection points.* Note that this considers only real numbers, and it's somewhat simplified relative to the way mathematicians think of roots of polynomials. Also note that none of it applies to functions other than polynomials. (See the bottom of this document for a comment on how this applies to *antiderivatives* of polynomials.)

Below is the graph of a "typical" cubic function, $f(x) = -0.5x^3 + 3x$, in blue, plus:

- its 1st derivative (a quadratic = graph is a parabola, in red);
- its 2nd derivative (a linear function = graph is a diagonal line, in green); and
- its 3rd derivative (a constant = graph is a horizontal line, in orange).

(The graph is also available—in color, of course—at http://www.informatics.indiana.edu/donbyrd/Teach/M119WebPage/Cubic+3Derivs_2P2N21.png .)



But why bother with the *3rd* derivative? Because it's the *2nd* derivative of the quadratic! So (ignoring the blue curve) the graph can also be interpreted as a "typical" quadratic function, $f(x) = -1.5x^2 + 3$, in red, plus:

- its 1st derivative (a linear function = graph is a diagonal line, in green); and

- its 2nd derivative (a constant = graph is a horizontal line, in orange).

1. A polynomial of degree *n* has at most *n* roots. For example, cubics (3rd-degree equations) have at most 3 roots; quadratics (degree 2) have at most 2 roots. Linear equations (degree 1) are a slight exception in that they always have one root. Constant equations (degree 0) are, well, constants, and aren't very interesting.

2. The first derivative of a polynomial of degree *n* is a polynomial of degree n–1, and its roots are the critical points of the original polynomial. The 2nd derivative has degree n–2, and its roots are the *potential* inflection points of the original polynomial. Therefore a polynomial of degree *n* has at most n–1 critical points and at most n–2 inflection points. In fact, most polynomials you'll see will probably actually have the maximum values.

3. For example, the 2nd derivative of a quadratic function is a constant. This means that a quadratic never has any inflection points, and the graph is either concave up everywhere or concave down everywhere. Does that make sense? Yes: the graph of a quadratic is a parabola, either opening upward or downward! For example, the 1st derivative of $f(x) = 5x^2 + 2x - 1$ is $10x + 2$. The 2nd derivative is simply 10, indicating concave up, for all values of *x;* and indeed $f(x)$ is concave up everywhere—and its critical point is a local minimum.

To summarize, for polynomials of 4th degree and below:

Degree	*Max. roots*	*Max. critical points*	*Max. inflection points*
4 (quartic)	4	3	2
3 (cubic)	3	2	1
2 (quadratic)	2	1*	0
1 (linear)	1*	0	0

(* = An equation of this degree *always* has this many of the this feature.)

Now, check your understanding. In the graph, do you see why the derivative of the red line (the quadratic) is the green line (linear)? Is the numbers of roots, critical points and inflection points for the cubic what you'd expect? How about for the quadratic, linear function, and constant?

What about Integrals?

Since *integration* (finding an integral) is the inverse operation to *differentiation* (taking a derivative), the graph might also help you understand the relationships between polynomials and their indefinite integrals or antiderivatives. Looking at it this way, we have the graph of a constant, a horizontal line, in orange, plus

- an antiderivative of it (a linear function = graph is a diagonal line, in green);
- an antiderivative of an antiderivative (a quadratic = graph is a parabola, in red); and
- an antiderivative of that (a cubic function, $f(x) = -0.5x^3 + 3x$, in blue).

Note: The cubic function and its derivative appear in Hughes-Hallett et al. *Applied Calculus,* 4th ed., Sec. 2.2, problem 21.

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