

Thoughts on Teaching

What I Learned about Why My Students Didn't Learn More

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May 2012; revised early February 2013

Background: Where I'm Coming From

I originally wrote this essay in May 2012 as the final paper for the final course for my M.S. in mathematics education, a degree I got as part of my late-career retraining as a math teacher. For my background through May 2010, when I began the fellowship, see my Brief (general) Autobiography (Byrd, 2010a), plus specialized autobiographies I wrote for various education courses: my Mathematical Autobiography, Intellectual Autobiography, and Cultural Autobiography (Appendices A through C). However, this version is intended to stand alone; it also has many improvements.

My Career as an Educator and the Evolution of My Thinking

I spent many years working in industry as a technologist, designing software, soundware (for music synthesizers), and user interfaces, and developing, maintaining, and documenting software. I returned to academia in the mid-1990's, mostly as a researcher but (at IU Bloomington, starting in 2003) also doing some teaching. But, starting in about 2006, I had more and more difficulty with funding for research, and teaching opportunities started to fade away around the same time. Just as my financial situation became serious, I found out about the Woodrow Wilson National Fellowship Foundation and its plans to offer fellowships to professionals in the STEM disciplines to retrain as teachers. I'd always loved mathematics, but had never really concentrated on it. In addition, I'd come to believe the average American's general lack of understanding of math was a very serious problem and wanted to help improve matters. So, early in 2010, I applied for a Woodrow Wilson Indiana Teaching Fellowship to retrain as a secondary-school math teacher. The entire application process went very smoothly. I was offered a fellowship to attend Indiana University/Purdue University at Indianapolis (IUPUI), accepted it, and began training in June of that year.

Naturally, the application process involved a face-to-face interview. But the interview had some unusual features, one of which was breaking applicants into small groups, with each applicant doing a five-minute "sample teaching lesson... on any subject" to a "class" of the interviewer and the rest of their group.

At this point in my life, I had several years of teaching behind me, though only part-time. But more important, all of my teaching had been at the college level, nearly all to graduate students. And all of it was in music, information technology, or some combination of them, not mathematics (though a significant amount of math was involved). Preparing this five-minute lesson was the first time I'd ever thought seriously about teaching math, and the first time I'd ever thought seriously about teaching *anything* at a level below college! I called my lesson "Zeno's 'Achilles and the Tortoise' Paradox vs. The Infinite Series" (Appendix D).

This paradox, Zeno's first paradox of motion, involves a race between Achilles (a very fast runner, said to be "the fleetest of foot of all mortals") and a turtle (a rather slow crawler), with the latter having a head start. Zeno gives a simple but surprisingly convincing argument that Achilles would never catch up: "In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead." In my brief lesson, I explained that this argument assumes that the sum of an infinite geometric series cannot be finite, and showed why that's not true, i.e., why such a series can *converge* (to use the technical term) to a finite value.

Psychological Theories and My Philosophy of Teaching

Motivation. Why, in my interview lesson, did I teach something as advanced as convergence of infinite geometric series? In two words, *intrinsic motivation*. I wrote "My Philosophy of Secondary Teaching" (Appendix E) some time ago, before my most important experience as a teacher, and it's really more about methods than philosophy; still, it should make it clear that I'm very concerned with motivation. High-school Algebra II invariably covers geometric series and their sums (Indiana Dept. of Education, 2006c), and infinite geometric series are very likely to be touched on. Still, students won't be expected to understand *why* the sum of infinitely many numbers can be finite until they take calculus. But, while the details require methods of calculus, the *concepts* are simple; there's no reason they can't understand immediately. In spring 2011, I took N517, Secondary Mathematics Methods, taught by Prof. Crystal Hill Morton, and one of the first assignments was to write a "Mathematics Autobiography". As mine (Appendix A) says, "I'm fond of the infinite, that fruitful source of crazy ideas", and I've wanted from the beginning to expose students to the wildest and craziest ideas I could possibly get them to understand. And why did I want to do that? The main reason is that, based on my own experience, I believe that wild and crazy ideas can be a great source of intrinsic motivation for a lot of kids—certainly not the majority, but a lot. And intrinsic motivation is generally agreed to be far more reliable than external motivation (Middleton & Jansen, 2011, p. 27). Along the same lines, Sawyer (1982, p. 143) comments "A non-mathematician learning mathematics...often has to plough through routine procedures, which can be extremely dull... An education should also contain elements that perform the functions of a cold bath—to provide a shock and keep one awake."¹

I ended my Mathematics Autobiography with the statement "I plan to get high-school kids *excited* about math. Or at least to try." In accordance with these views, ever since I started thinking about teaching math, I've been compiling a list of "Math is Cool, Fun, Wild, etc." Teaching Ideas (Byrd, 2012d). I'll have more to say below about my efforts thus far to get kids excited about math.

A critical aspect of motivation my "Philosophy" document addresses is what is commonly called laziness. As it says, "I've never known anyone, student or not, that I was convinced

¹ This passage occurs at the beginning of a chapter on projective geometry. In the next paragraph, Sawyer comments that projective geometry "abounds in beautiful impossibilities." The idea of "beautiful impossibilities" in mathematics is striking because, for me, the fascination of math lies primarily in the way it shows "obviously" impossible things to be possible; but it seems that what a great many people who love mathematics (e.g., Douglas Hofstadter, personal communication, April 2012) find most attractive about it is its beauty. Lockhart (2009, p. 92) describes "Mathematical Reality" as "breathhtakingly beautiful and entrancing", and says quite a bit more about why he loves math.

was lazy. Laziness sounds like a fundamental aspect of a person's character. But how can you know that, especially about someone you've known only in the context of school? A much better way to think about someone that doesn't want to work on whatever is at issue is just that they're unmotivated... It may not be possible to motivate students to do what you want them to do, but once you decide they're lazy, you're already most of the way to giving up on them." I'll have more to say about so-called "laziness" later on.

Knowledge, thinking, and the psychology of learning. I find it difficult to take behaviorism (Phillips & Solits, 2004, Chapter 3) seriously as a theory of almost anything. Flanagan (1991) comments about B. F. Skinner, certainly the best-known advocate of behaviorism, that he "tends to throw the baby out with the bath water and make psychology epistemologically sound at the price of making it epistemologically impoverished". I agree, and the price is so high that behaviorism has very little to offer to education. Cognitive information processing (CIP) (Bruning et al, 2004, Chapter 4) is much more interesting. Its acceptance of mental processes as relevant opens the door to considerations like the limitations of short-term memory. Bruning concentrates on practical applications and says little about theory, but the theoretical claims of CIP also seem entirely consistent with what I know of cognitive science (Minsky, 1988; Hofstadter, 2007). But, to my mind, much of the importance of CIP is that it's a step towards constructivism.

For me (as well as, apparently, a very large number of educators and researchers), constructivism is incomparably the most far-reaching and important theory of learning; see Driscoll (2005, Chapter 11) and Phillips & Soltis (2004, pp. 41–52). Holt (1982) offers more insight into why teaching (and not just of math) is difficult than anything else I've ever read. Among his deepest insights (p. 145) is that "I doubt very much if it is possible to *teach* anyone to understand *anything*, that is, to see how various parts relate to other parts, to have a model of the structure in one's mind. We can give other people names, and lists, but we cannot give them mental structures; they must build their own." This is as clear a statement as I can imagine of the constructivist view of understanding and mental models.² His comments on superficial evidence of understanding ("blip blopping", p. 139–140) alone are worth far more than the price of the book. Holt relates his insights directly to his own experience teaching elementary school (5th grade, I believe), but they apply almost as clearly to secondary school.

Holt's book is largely a journal of his experiences in the classroom, something that precludes an organized approach to describing things. Perhaps the closest thing to a systematic statement of how (in my view) one *should* approach teaching math at the high school (and lower-level undergraduate) level is G. Polya's (1957) classic work *How to Solve It*. Polya strongly emphasizes the Socratic method, i.e., leading students to discover things for themselves; what makes the book so valuable is that he gives many detailed examples of

² At one point well along in my former career as a software engineer, I came across an introductory article on the programming language Smalltalk written by a well-known authority on it. I was very pleased because I'd long wanted to understand Smalltalk's basic ideas. However, I never finished reading the article. I found it incredibly hard going because it repeatedly referred to features of Smalltalk that that sounded a lot like features of popular languages I knew well; but it used different terms to refer to them, and it never said whether Smalltalk's features really *were* similar to the familiar constructs or not. In fact, it never said anything at all about the relationship between the Smalltalk constructs and the familiar ones! I tried to ignore the question, but it nagged at me more and more, and I finally gave up. Why? The article effectively asked me to build up an elaborate cognitive structure that closely resembled part of an even more elaborate structure I already had in my mind, without giving me even a clue as to how to relate the two.

how to do it, with insightful comments (including some about how *not* to do it). (Polya claims his book applies to problem solving—and to teaching problem solving—in many areas outside of math, and I agree.)

Most of the sources I've cited are rather old. One recent source that relates the theoretical and research foundations of education to teaching mathematics is Hiebert (2003); he focuses narrowly on relating research to current teaching standards.

Cognition, perception, and computation. I've long believed that—almost without exception, and regardless of their background—people grossly underestimate the complexity and subtlety of basic aspects of human thought, including perception as well as cognition. This is by no means an original idea; one proponent is the well-known cognitive scientist Douglas Hofstadter. He has argued that the equally-well-known Herbert Simon is one of the underestimators (Hofstadter, 1985, p. 632):

In 1980, Simon... [declared] "Everything of interest in cognition happens above the 100-millisecond level—the time it takes you to recognize your mother." Well, our disagreement is simple; namely, I take exactly the opposite viewpoint: "Wrong. Everything of interest in cognition happens *below* the 100-millisecond level—the time it takes you to recognize your mother." To me, the major question of AI is this: "What in the world is going on to enable you to convert from 100,000,000 retinal dots into one single word "mother" in one tenth of a second? Perception is where it's at!

In other words, the interesting aspects of cognition are precisely those that are considered to be perception and "hidden" (because they happen too quickly). I agree with Hofstadter, and in my view, many of the pedagogic mistakes Holt and others describe are direct consequences of misconceptions of this type. Along the same lines, researchers of Simon's mindset have built models of intelligence based on explicit symbols, and written about "cognition as computation" (Barr, 1982). Hofstadter (1985, p. 643) argues that the remarkable flexibility of our minds cannot be produced by such machinery; that symbols are epiphenomena, that is, that they are visible manifestations of the organization of the brain and are not explicitly present; and that "not cognition, but subcognition, is computational."³ Again I agree, and I believe the educator's assumption (quite likely unconscious) that explicit symbols exist in the mind leads them to the questionable assumption that they can create the "same" symbols in students' minds in a straightforward way.

Classroom Experience

To date, I've taught middle school and high school for one semester each (as a student teacher); 100-level calculus at a college for a semester (just one section); and high school as the teacher of record for six weeks, as a maternity-leave replacement. Here's an overview of what happened in those classrooms. The maternity-leave replacement position is most relevant for several reasons—it's the most recent as well as by far the most concentrated experience—so I'll cover it at much greater length than the others.

Note that lessons and activities I describe as "most successful" below are largely those in which I felt students were especially engaged, and those in which they did the largest amount of independent thinking. These are, of course, not necessarily the factors the average teacher or administrator would consider most important!

³ Hofstadter (1985) is intended to stand alone, and is also a good source for definitions of the technical terms I use.

Secondary Teaching as a Student Teacher (Fall 2010 and Spring 2011)

Middle School

In the fall of 2010, Lydia Sitnikov and I were student teachers in the 6th-grade classroom of Amber Krieger at Northview Middle School in Washington Township on the northeast side of Indianapolis. In accordance with the IUPUI teacher-training program Gradual Release of Responsibility, Lydia took over more and more of leading three of Ms. Krieger's classes, while I did the same for her three other classes: two of regular 6th-grade math, and one honors class (which followed the regular 7th-grade curriculum). However, each of us assisted the other in various ways. The school used a block schedule, with teacher teams.

I had a great deal of trouble preparing for classes adequately. Finally Amber Krieger (my mentor teacher) and Dwight Schuster (site supervisor) called a meeting with me and Kathy Marrs; the upshot was a written "Improvement Plan for Fellow Donald Byrd", signed by all of us. This plan largely solved the problem, though I continued to find preparation difficult.

Most successful lesson/activity: two lessons for the honors class on indirect measurement. For one lesson, I used masking tape to create 20-ft. long x and y axes on the floor of a large room, then had five students at a time do a "Similar Figures and Scale Lesson" (Appendix F) involving a tape measure and a laser pointer. Briefly, under my supervision, the Director had the other four students position themselves at the vertices of an 8 ft. by 5 ft. rectangle, with one vertex at the origin. Then the Director rolled a die to determine a scale factor and calculated the new x and y distances. Leaving the student at the origin where they are, the Director had the other students move to the three new vertices, using a tape measure, the x and y axes, and (for the diagonal) the laser beam. The second lesson involved using a flashlight to measure the lengths of shadows cast by a tall student, a short one, and a ladder. Then, given the heights of the students, they had to write and solve a proportion for the height of the ladder.⁴

High School

I spent my second semester as a student teacher in the classroom of Paul Weedling at Crispus Attucks Medical Magnet High School, one of the Indianapolis Public Schools, in fact just a few blocks from IUPUI. I taught two sections of precalculus.

The teaching environment in Mr. Weedling's room was extremely challenging. The school didn't provide students with textbooks, either individual copies or a classroom set. Also, it being a magnet school, many students—perhaps 30% of my classes—took courses at IUPUI, with the administration's blessing. That might have been a good idea, but their schedules went in and out of sync with ours because of our block schedule, so they often missed entire weeks.⁵ In response, I created supporting materials, including 2-column notes for each lesson, and put them online; the final version of the class website is at

www.tinyurl.com/DABPrecalc

⁴ While both lessons were reasonably successful, they undoubtedly would have worked better if, for the first, I'd been more realistic about the amount of time needed, and, for the second, if I'd remembered to bring some good tape to keep the flashlight in place.

⁵ In theory, textbooks weren't necessary because every student was supposed to have a school-issued notebook computer with access to materials online; the reality was otherwise. It seems incomprehensible that the administrators responsible would have allowed this situation to arise, especially the scheduling conflict. I can only say that the Indianapolis Public Schools do not have a reputation for being well-run.

I'm not certain if this resulted in students doing better, but it did make it possible for them to take more responsibility for their learning; it weakened their excuses for not learning.

Most successful lesson/ activity: Following Mr. Weedling's lead, I had my students do *algebra aerobics* to help them understand systems of equations: "Have students imitate graphs of equations with their arms, following teacher (e.g., $y = x$; $y = -x$; $y = x^2$; $y = x^2 - 1$; $y = (x - 1)^2$; etc.). Then students pair off, each facing their partner, and show systems of two equations with specified numbers of solutions (1, 2, 0, 4, an infinite number): the points where their arms intersect represent solutions." This was a huge success. Later, they were having a hard time getting used to measuring angles in radians instead of degrees. I guessed something similar I called *angle aerobics* ("Turn 90° ; turn $\pi/4$; turn π ; turn -540° ; which way should you be facing now?") would help. It did.

College Teaching as a Part-time Instructor (Fall 2011)

During fall semester 2011, I taught one section of M119, Brief Survey of Calculus, at IUPUI. The only degree that requires M119 is in the School of Business, and, not surprisingly, most of my students were or hoped to become business majors. Many of them found calculus difficult. Encouraged by what I saw as the value of the website I'd created for my spring precalculus classes, I created a website for this class as well:

www.informatics.indiana.edu/donbyrd/Teach/M119WebPage/CalculusInfo.html

It can also be reached with this shortcut:

www.tinyurl.com/DAB-M119

Wanting to help my students as much as possible, I also offered many extra-credit options. This was my first time as teacher of record for a math course, and therefore my first time giving course grades. Results were fair; I had 27 students initially; 4 dropped the course, and of the 23 that finished, 3 failed and 2 got D's, for a "DFW" rate of 33% as compared to the typical rate of 30%.⁶

Most successful technique: The classroom—which had room for several times as many students as I had—had six small whiteboards on the sides. I assigned students to small groups, each with at least one of the best students, and had each group go to a whiteboard and work together on a problem (the same problem for all groups). Then we all talked together about what they'd come up with.

Secondary Teaching as a Maternity-Leave Replacement (Spring 2012)

The School and Context

I finally began a position as a regular classroom math teacher, albeit a temporary position, this January. The job, at Brownstown Central High School (BCHS) in Brownstown, Indiana, was replacing Mrs. Andrea Pendleton, who was going on maternity leave for 10 weeks. Mrs. Pendleton has been teaching at BCHS for over 10 years. The principal told me she's quite possibly the best teacher in the school; I observed her for a full day, and that's not hard to believe. BCHS is on the trimester system. I took over for the last six weeks of the second trimester of the school year, and was to teach through the first four weeks of the third trimester.

⁶ This "typical rate" is according to John Miller, the course coordinator and a long-time lecturer in the IUPUI Department of Mathematical Sciences.

A BCHS school day has five 70-minute periods; with one for prep, teachers have just four classes. Mrs. Pendleton's were all honors classes, and for the second trimester, they were one each of Geometry trimester 1, Geometry trimester 2, Algebra II trimester 1, and Algebra II trimester 2: only two subjects, but four preps. When I observed Mrs. Pendleton teaching, I saw essentially no behavior problem. This is what I expected from honors classes. Still, I knew the four preps alone meant this was going to be a tough job for me as a first-year teacher, and I knew Mrs. Pendleton would be a hard act to follow (as the expression goes). Fortunately, I had a "support system", namely John Buckwalter as IUPUI liaison person, plus the teacher next door: Mrs. Warren, the chair of the BCHS Math Department. She was knowledgeable (she'd also been at BCHS for over 10 years) and friendly, and John arranged for her to be my official in-school mentor.

What Happened

On my first day, I handed out a "Letter to Parents and Student Information Form", emphasizing my availability and my intent to do things the way Mrs. Pendleton did as much as possible, and I told the students a few things about myself and my family. I decided from the beginning to cultivate an eccentric and playful persona in front of my classes; this was easy to do because I really am both somewhat eccentric and decidedly playful! For example, I told them we have a dog named Antidisestablishmentarianism, and I explained where the name came from (the reason has minor mathematical implications⁷). And I told them that we'd have some fun and learn something every day, though not necessarily in that order.

The main thing I wanted to accomplish with all of this was to connect with students, and I feel it worked quite well. As evidence, after two or three days, students started writing (and signing) messages to and about me on the greenboard at the front of the classroom, messages like "Mr. Byrd is awesome"; "We love you, Mr. Byrd"; "Your first week was great. :-) Keep it up! We [HEART] u!"; "You ROCK! [HEART]", etc. These messages—each with two or more names at the bottom—died out after the first couple of weeks, but for the rest of my time at BCHS, students (including some I didn't know at all) went out of their way to say hello to me in the hall.⁸

Most successful lessons/activities: (1) The "Monty Hall problem". One of my Algebra II classes did a unit on probability theory. I've found probability to be *the* most unintuitive subject I've ever studied as well as one of the most important, and I decided it was worth supplementing the textbook material with a lesson on the justifiably famous "Monty Hall Problem", a puzzle that got national attention as a result of Marilyn vos Savant's articles about it in *Parade Magazine* and subsequent attacks on her by a large number of people, including several mathematicians. But vos Savant was right from the beginning. (One good source of details is her columns (vos Savant, 1990a, 1990b, 1991); they appear, complete with multiple attacking letters her and her responses, in vos Savant (2006). The Wikipedia article

⁷ The name was inspired by the children's song, Bingo: "There was a farmer had a dog and Bingo was his name..." It has one verse per letter of the dog's name, and the length of each verse is also a function of the number of letters in the name. This suggests the question, how much longer would it take to sing if the dog's name was a great deal longer—say, 28 characters instead of 5?

⁸ One day around halfway through my weeks at BCHS, recruiters for the Marines visited, and they'd set up a frame in the cafeteria to test student's ability to do pullups. When I walked into the cafeteria that day with my lunch tray, I was greeted with calls of "Mr. Byrd! Mr. Byrd! Do it!" One of my avocations is rock climbing, which demands much upper body strength, and I was able to do 10 pullups; that clearly made a big impression on many students.

“Monty Hall problem” is well-written and goes into great detail. For a brief, mathematically rigorous discussion, see Grinstead & Snell, 2007, pp. 136–139.) Mimicking the situation on the TV game show “Let’s Make a Deal”, the question is whether, according to the laws of probability, the contestant should stick with their initial answer to a certain question or switch. It turns out that switching doubles their chances of winning, yet an overwhelming majority of people—even those with very significant mathematical training, and even some research mathematicians—conclude it makes no difference.⁹ The mathematics involved is actually rather simple. The difficulty is entirely a matter of thinking clearly about what is known and what affects the outcome.

I divided my 19 students into small groups and asked them to discuss the situation for a few minutes, then say what they thought the answer was. The result: some thought it was better to stick, and some thought it didn’t matter; but not a single one realized the contestant should switch! The lesson engaged every student, and I’d go so far as to say several were fascinated.

(2) Indirect measurement. I had two students use a laser pointer and protractor with a bit of trigonometry to determine the length of the classroom from its (known) height; then I had every student roughly check it via the “Rule of Thumb”. Again, every student seemed to be engaged.

Problems with Parents. On the other hand, the principal told me in about my 4th week that, while he knew I was very friendly to students and worked very hard, he’d received many complaints from parents about their children not understanding what I was teaching and their grades going down. In contrast, he’d gotten *no* complaints about Mrs. Pendleton for many years! Of course, Mrs. Pendleton is an outstanding teacher, and I have no doubt that complaints about me resulted to a considerable extent from my inability to teach as well as her. But I believe they were also caused in part by my not teaching quite the way she did, not knowing the best way to handle parent complaints, etc. But, while these are important matters, they’re really matters of politics, not teaching.

The End. The principal told John Buckwalter that, in view of parents’ complaints, it might be good if I resigned at the end of the second trimester, i.e., after my sixth week. This surprised him as well as me, since it’s generally not easy to find a teacher to step in in the middle of the school year, and especially one who can teach honors students. But it turned out that the principal had a teacher he knew and liked ready to step in: a retired math teacher, in fact a man who’d been Mrs. Pendleton’s and Mrs. Warren’s teacher! In any case, I took the suggestion and quit early. My last day at school, I put “Math is Wild” (Appendix G) on the screen; it should be obvious from it that my goals as a teacher have changed not at all from my goals as of my interview for the Woodrow Wilson fellowship.

I’d describe my work with students at BCHS as uneven: very successful in some ways, not too successful in others. Did I get any students *excited* about math? I definitely think so. Did I teach the average student as much as Mrs. Pendleton would have? That’s the wrong question; see the section “students really are responsible for their own learning” below.

What I Learned about (mostly Secondary) Teaching

⁹ In fairness, the odds of winning depend critically on details that are often left unstated. I was careful to give my students these details. (In addition, studies suggest that—even when the details are omitted—most people make assumptions that lead to the probabilities I’ve given.)

Of course, much—probably most—of what I’ve learned about teaching can’t easily be put into words. I feel my intuition about what to do next in any given situation has gotten better and better, though it’s still nowhere near that of any competent teacher with, say, 10 years in the classroom! But here are a few concrete things. It’s important to note that in some sense I knew most of these things very early, certainly after my first month of student teaching; but I didn’t know any of them (with one possible exception) *deeply* enough to apply them *effectively*. By the time I started this essay, 18 months later, I could apply all of them. The education courses I took were useful, but I learned more from experience in the classroom—and I learned a great deal from outside reading, especially the superb books by Polya (1957), Holt (1982), and Lockhart (2009), the latter with its scathing and eloquent critique of math-education practice.¹⁰

Books in the NCTM series *Empowering the Beginning Teacher in Mathematics* apparently include “Top 10 things I wish I had known when I started teaching” lists; it would be interesting to compare these lists to my 11 items. NCTM’s “blurb” for the books observes, “Not every student will be interested every minute. No matter how much experience you have or how great you are at teaching, you will encounter times in the classroom when no student is interested! Find the solution to this classroom dilemma and read more about the other nine ‘things’ in the series, *Empowering the Beginning Teacher in Mathematics*.” (<http://www.smartbrief.com/servlet/encodeServlet?issueid=0D0D7604-FABE-42B6-A0A8-D644C00386DF&sid=723cf31a-2445-43d2-8186-0062a1257c9e>)

Here’s my list. Naturally, these ideas don’t apply equally to students of all ages; in general, the younger the students, the more relevant they are.

1. To a great extent, students really are responsible for their own learning. To be crystal clear, I mean that students are in control more than anyone else is—certainly more than their teacher—of how much and what they learn. This may be the single most important thing I learned. This idea can be paraphrased in many ways, for example, a statement that’s often attributed to Einstein: “I never teach my pupils; I only attempt to provide the conditions in which they can learn”¹¹, and Holt’s statement beginning “I doubt very much if it is possible to *teach* anyone to understand *anything*”, quoted under Psychological Theories and My Philosophy of Teaching above. But I’ve tended to feel that, if the students aren’t learning, it’s “my fault”, and one day during the semester I taught calculus at IUPUI, I said something to a far more experienced instructor that made that clear to her. She told me in no uncertain terms that it was a mistake to have that attitude, but I still didn’t get it. Near

¹⁰ Lockhart’s is the best critique I’ve ever seen of contemporary math education in the U.S. It seems to be fairly well-known to college math instructors, but—ironically—not to secondary-school teachers and teacher educators; in fact, I’d never heard of it until a few months ago. The book form includes both the “Lament” proper (his critique), written in 2002, and a more recent “Exultation” on “what math really is and why I love it so much”, which is also quite valuable. The downloadable .pdf listed in the References contains only the critique.

¹¹ I don’t know an authoritative reference for this fascinating quotation. Google finds tens of thousands of matches for Einstein’s name with this phrase. Needless to say, I looked at only a few, and only one gave any additional information (Mazer 2011): “Earliest published source I find is the 1968 book *Training within the organization: a study of company policy and procedures for the systematic training of operators and supervisors* which on p. 126 says: ‘It was probably in the latter sense that Professor Einstein in talking about teaching once remarked: ‘I never teach my pupils. I only attempt to provide the conditions in which they can learn.’” No source is given, and none of the other books I saw gave a source either.”

the end of my time at BCHS—probably around the beginning of the fifth week—a class went especially poorly one day, and I thought of five or six reasons why. All or nearly all were things I’d done or failed to do (one obvious one was that I hadn’t tested a piece of technology I was counting on and it didn’t work). The next day, before that class, I talked to Mrs. Warren about my frustration that students weren’t “getting” what I was teaching, and that very few were taking advantage of my repeated offers of help outside class. To my amazement, she told me that Mrs. Pendleton typically had 10 or 15 students in her room getting help before school! Hearing this, I suddenly realized that my teaching was by no means the only reason my students were having trouble. I might have one student come in before school and one after school, and often not even that. But clearly they weren’t coming in outside of class because they weren’t motivated; wasn’t this in itself a problem with my teaching? It’s hard to tell, but—being as honest as possible—I’d say it was mostly because of factors that I couldn’t control, for example, the temporary nature of my position. (I told my class what Mrs. Warren had said and again urged them to get help from me outside of class; I didn’t see any increase in help seekers, but I felt considerably better about my teaching after that.)

Thus, the question above “Did I teach the average student as much as Mrs. Pendleton would have?” is seriously misleading. A better question is “Did the average student learn as much as they would have with Mrs. Pendleton?” The answer is almost certainly no, but there were mitigating circumstances.

2. The appropriate preparation for a class varies greatly with students’ age, maturity, etc.

As a student teacher in middle school, I discovered over and over that I hadn’t prepared thoroughly enough (resulting in my aforementioned “Improvement Plan”). In particular, transitions are an opportunity for 6th-graders to get completely distracted; a transition I didn’t manage well could easily waste five minutes of class time. I found my best bet was to have a written plan for every lesson describing what I’d do in some detail, including transitions (Kaser 2007). But high-school students—honors or not—are considerably less distractable, and generally have no problem with transitions. With them, I found I was better off spending less time on detailed plans and more time on other things, e.g., having written solutions to the more difficult homework problems. That really made a difference. (One reason I was less effective than Mrs. Pendleton was that she could answer almost any question a student might have immediately; I could answer *many* questions immediately, and *most* questions *almost* immediately.)

3. When presenting content, the more concrete, the better. This is something I thought I knew quite well by the time I started teaching at BCHS, but I found otherwise! For example, many of my Algebra II Honors students were having trouble understanding negative exponents. To make the relationships involved more concrete, I drew the following table on the greenboard, showing how positive integer exponents relate to repeated multiplication:

x^1	x^2	x^3	x^4
x	$x \cdot x$	$x \cdot x \cdot x$	$x \cdot x \cdot x \cdot x$

Then I observed that each time you add one to the exponent, you multiply by x one more time. I asked the class how this could be extended to non-positive exponents. Not getting an answer from them, I answered myself by pointing out that that rule is equivalent to the rule that each time you *subtract* one from the exponent, you *divide* by x one more time. I added a few columns to the left:

x^{-2}	x^{-1}	x^0	x^1	x^2	x^3	x^4
$1/(x \cdot x)$	$1/x$	1	x	$x \cdot x$	$x \cdot x \cdot x$	$x \cdot x \cdot x \cdot x$

I commented further that defining powers of x below the 1st that way is the only way to be consistent with positive powers.¹²

All of that helped some, but not as much as I expected. It was only much later that I realized I could easily have made things *really* concrete simply by substituting, say, 3 for x . (It might also have helped if I'd used a variable other than x , since x looks a lot like a multiplication sign.) I've made this mistake of not being as concrete as possible many times. I think the main reason is that I tend to assume that expressions like the ones above are already so concrete, there's no need to get even more so. But that's not a safe assumption, to put it mildly—not even with an honors class like mine.

4. In classroom management, the more concrete, the better. Many of my 6th-grade students had a hard time controlling themselves, and one of my standard tactics was to make a list of offenders I would make stay after class. A very common problem, and, no doubt, a common way to handle it. But it was obvious that the clearer the connection between the behavior and the consequences, the more effective a deterrent it was. Lydia and I started putting a timer on the smartboard to keep track of how much time of ours the offending students had wasted, and announcing that they'd be staying after class for the same length of time. To make the connection even more concrete, I looked for a simple, easy-to-use program that would count down whatever length of time it had gotten up to, and preferably one that could run in a Web browser with no installation. But, at the time, I couldn't find one: every program I ran across treated going up and down as unrelated functions, typically calling them "stopwatch" and "timer". (With my experience as a software engineer, I finally wrote such a program myself, "Don's Up/Down Timer"; it's now available for all to use (Byrd 2012e).)

5. It's important to have more than one way to explain anything. At least one parent complained that, when their child told me they didn't understand something, I simply repeated what I'd already said. This reminds of a comment by a college geometry instructor to his class of (mostly) future high-school math teachers: it's important for a high-school geometry teacher to be "light on their feet", i.e., to be able to change course quickly based on students' understanding or lack of it (R. Patrick Morton, personal communication 2011). I wasn't light on my feet.

¹² A great deal of the development of mathematics since ancient times can be explained by the desire to extend methods that apply in an obvious way in some circumstances to wider and wider ranges of conditions. For example, exponentiation was originally defined for rational-number bases and natural-number exponents: say, $(3/4)^5$. Then it was extended to real-number bases with, first, any integer, then any rational number, then any real number as exponent: expressions like $e^{-2.5}$ are not unheard of. We now allow raising complex numbers to complex exponents. But, of course, care must be taken that any such extension is defined in a way that's consistent with previous definitions; if not, the consequences are likely to be disastrous! This basically historical approach can definitely make things more concrete for students, and it has often been used in math education: see, e.g., National Research Council (2000), p. 166. Katz (2009) includes an appendix on "Sample Lesson Ideas to Incorporate History" (pp. 935–939) that describes four approaches to using the history of mathematics in teaching, two of which approximate what I'm talking about. Sawyer (1982, p. 144) offers very thoughtful comments on the advantages and disadvantages of a historical approach to teaching geometry.

6. Technology can help, both inside and outside class, but be careful with it. Each of my four classrooms had a computer projector. In the student-teaching rooms, it functioned as a smartboard; at IUPUI and BCHS, I just used it to show the screen of a computer, for example to show Geometers Sketchpad demos of geometric phenomena and graphs of functions. In addition, *Tips and Resources for Solving Math Problems* (Byrd 2012a) referred students to a number of resources they could use on their own, especially the Khan Academy and PurpleMath websites. Many students told me they found these helpful. On the other hand, in my first weeks at IUPUI, I spent a fair amount of time showing my class graphs and other things on the computer without really connecting them to what we were studying, and I was justly criticized by more experienced teachers for wasting the time.

I believe computer technology has great potential for mathematics education, and I would have loved to have students actually use computer technology themselves, not just see me using it for demonstrations. I consistently looked for opportunities to do that, but I almost never found it practical.

7. Some students' problems are beyond anything a classroom teacher can solve. Here are two examples from my college calculus class.

(a) *Problem with course content.* "Colton" told me he always struggled with math, and indeed he did in my class. He spent as much time in my office as all the other students combined; that seemed to help some, but not that much. And he wasn't at all stupid. Like so much mathematics, calculus requires a fair amount of algebra, and I already knew many of my students' problems were more with algebra than with calculus. So I finally asked Colton a *very* basic question: How much is $3 \times (5 \times 4)$? I wasn't too surprised that he tried to use the Distributive Law, as if the question was $3 \times (5 + 4)$! Colton was a senior, and older than most—probably 23 or 24 years old, and the Distributive Law is generally taught by 6th or 7th grades (under the current Indiana standards, it's taught in both: see Indiana Dept. of Education, 2006a, 2006b). So he'd undoubtedly been confused about this for well over 10 years, and it takes a long time to unlearn something that deep-seated.

(b) *Problem unrelated to course content.* At the beginning of the semester, "Isolde" really seemed to want to do well, but she did several odd things that hurt her grades enormously. For example, she had a graphing calculator but not a non-graphing one. I didn't allow graphing calculators on quizzes/exams, so she started the first quiz/exam without any calculator until I noticed and loaned her one. Exactly the same thing happened on every other quiz/exam! She kept missing classes and even exams, and she never came to my office hours; she blamed all of that on transportation problems. And she repeatedly forgot important things. Perhaps she had tremendous family responsibilities, or she might have had ADD, or both. But she never told me what was going on, and, sadly, I couldn't find a way to help her.

Example (b) is clearly related to my item #1, "students really are responsible for their own learning"; but example (a) is something else.

8. It's important to be realistic about your strengths and weaknesses and adapt accordingly. An important special case: Grading homework and exams can easily overwhelm the teacher. For example, by the time I started at BCHS, I knew I was a slow grader. Preparing four lessons every day took so much time that I quickly realized that I'd have very little time to grade homework, and from the beginning I didn't try to grade as much homework as Mrs. Pendleton did. But I couldn't even do what I thought (and told my students) I could. It wasn't until I graded the final exam that I realized *how* slow I was! I can see several reasons for this: wanting to give useful feedback, wanting to grade as consistently as possible, inexperience, etc. I expect to get more efficient, but not by so much

that it won't always be an issue. But the teacher usually has quite a bit of flexibility in how much grading they have students generate. One reason is that there are often ways to substantially reduce the load of grading without harming learning, so the teacher can avoid being a victim of a system they created themselves. Some ways that have been used successfully (Ofer Levy, personal communication, April 2012) are appropriate for teaching math: assign group instead of individual work; let students correct each other's work; go over homework and quizzes in class but don't collect or grade them.¹³ I did some of this, but could have done much more.

9. An element of discovery by students is tremendously valuable. In each of my teaching environments so far, the most successful lesson or technique has been the one, or one of the ones, that involved students to the greatest extent in finding things out for themselves. Of course, the argument that it's important for students to find things out for themselves has been made many times—and in elementary mathematics, as far back as the 1820's, when the American Warren Colburn published his first arithmetic textbooks (CMI 2012). We now describe such educational methods with buzzwords like "inquiry-based learning" (IBL) and "project-based learning" (Boaler 2008, pp. 68–74; Bressoud 2011a, Bressoud 2011b, AIBL 2012).¹⁴ Along the same lines, the distinguished mathematician Paul Halmos (1975) has written "The best way to learn is to do; the worst way to teach is to talk." But, despite their rich history, it seems that for many years very little attention has been paid to these ideas in math education in the United States; see for example Boaler (2008, Chapter 2) and Bressoud (2011a). Also consider the statement that "the basic [math education] goals of the United States apparently consist of learning terms, definitions, and computational processes" (Hiebert, 2003, p. 13). None of this involves students discovering anything. Still, articles and books by secondary-school and college instructors that emphasize the importance of math students finding things out for themselves have been appearing all along. Polya (1957) is a great example from the mid-20th century. The book begins with a section entitled "Helping the Student", in which he comments (p. 1)

The student should acquire as much experience of independent work as possible. But if he is left alone with his problem, without any help or with insufficient help, he may make no progress at all. If the teacher helps too much, nothing is left to the student. The teacher should help, but not too much and not too little, so that the student shall have *a reasonable share of the work*.

After introducing "an elegant and fascinating problem" in his wonderful book *A Mathematician's Lament*, Lockhart (2009, p. 123) goes so far as to say "The right thing for me to do as your math teacher would be *nothing*." But it's clear from the context that he means that in the sense of having a very long "wait time" to let students see what they can figure out on their own, and even then, to not help them too much.

Mathematics Teacher, a publication of the National Council of Teachers of Mathematics, is one good current source of articles describing specific lessons with major elements of inquiry by student; see for example Nabb (2010).

¹³ Another technique Levy suggested is: limiting space available for students to fill in answers. This might be wonderful for some subjects—say, social studies—to discourage students from giving excessively verbose answers. But it does not seem like a good idea for math, where it would likely discourage students from showing their work: something that no math teacher I know wants to do.

¹⁴ IBL seems to have been re-introduced into mathematics education at the college level, with the work of R. L. Moore as far back as the 1920's.

10. One-tenth of a picture is worth 100 words. By “one-tenth of a picture”, I mean considering the graphic aspects of tables, formulas, and even ordinary text. Obviously my epigram is based on the old saw, a picture is worth 1000 words. There’s much truth to that, including in teaching math, and I think math teachers are reasonably aware of it. But there’s less awareness of how valuable general graphic communication principles can be with non-pictures. For example, the standard mnemonic for order of evaluation of operators in expressions is “PEMDAS”: handle Parentheses, Exponents, Multiplication, Division, Addition, Subtraction, in that order. I taught this to my 6th-grade students, just as my mentor did. But I was not surprised to find many of them were confused by the fact that multiplication and division are “on the same level”, and likewise addition and subtraction. Then it occurred to me to show the grouping explicitly by writing “PE(MD)(AS)”, and that helped considerably. Along the same lines, using color, boldface, etc. to emphasize particular values in tables can make them much easier to get useful information from.

11. There are no lazy students, only unmotivated students. To quote again from my Philosophy of Teaching, “Laziness sounds like a fundamental aspect of a person’s character. But how can you know that, especially about someone you’ve known only in the context of school? A much better way to think about someone that doesn’t want to work on whatever is at issue is just that they’re unmotivated. In response to my telling him this, one very experienced teacher commented: ‘Even the laziest people I have known were willing to work hard on things that interested them, and I suspect more than a few were depressed or discouraged.’ It may not be possible to motivate students to do what you want them to do, but once you decide they’re lazy, you’re already most of the way to giving up on them. That’s an easy out for the teacher.”

I wrote in the same document that I’d never known anyone, student or not, that I was convinced was lazy. My experience in classrooms these last few years hasn’t changed that, nor has it changed my feeling that this is a really important point. As just one example, consider an exceptionally talented student who doesn’t work in class simply because the material is too easy and they’re bored stiff! This is important both because it may well be more common than educators realize (how can anyone know?), and because it’s particularly unfortunate to lose talented students without the teacher even knowing that that’s what’s going on. Yes, little Ali Eisenstein might truly be lazy, but probably not. As her teacher, it’s not likely you’ll ever know for sure, and assuming she is has far more potential to hurt than to help.

Original Plan for a Y510 Project: Teach Constructible Regular Polygons

Under the initial plan for the Y510 course, I worked on “Using Higher Math to Improve Student’s Understanding”. I intended to show a high-school geometry class the close relationship between algebra and geometry, expecting this to improve students’ understanding of both subjects. Chapter 5 of Sawyer (1982), “On Unification”, comments:

“One of the most satisfying moments in mathematical history is the instant when it appears that two departments of mathematics, until then regarded as separate and unconnected, are in fact disguised forms of one and the same thing... One such striking piece of unification is within reach of the school syllabus. School mathematics seems to fall into two parts. On the one hand we have arithmetic, from which develops algebra, dealing with numbers. On the other, we have geometry and its development trigonometry... mainly concerned with shapes.”

After a page of discussion, Sawyer cites a “famous equation” (Euler’s Formula) that “brings about the complete annexation of trigonometry by algebra”.

The specific topic I planned to teach was “Constructible Regular Polygons”: constructible in the sense of early Greek geometry, where the only tools allowed were straightedge and compass, and they could be used only in certain ways. It turns out that constructing a regular polygon with these constraints is possible only if the number of sides of the polygon is a *Fermat prime*, a very unusual type of prime (only five are known to exist); a product of distinct Fermat primes; or a power of 2 times a product of distinct Fermat primes. This is a very surprising fact, one that mathematicians didn’t discover until the advent of abstract algebra in the early 19th century. My lessons would have been suitable in most schools for honors classes only. However, since I didn’t have a classroom of my own after mid-February, it was agreed that I’d write this paper instead.

Conclusions: My Future...

My Future As a Teacher

The late Jef Raskin, founder of the Macintosh project at Apple, was also a talented musician. While he’d composed quite a bit of music, he told me that he’d given up composing because he discovered that when he continued to work on music he had written, it didn’t get better. Well, I’ve worked seriously on writing music, technical papers and non-technical articles, and software, and on creating and teaching lessons. I’ve found over the years that when I work on my music, papers/articles, and software, they get better. With lessons, the evidence so far is that they *don’t* get better, and that makes me pessimistic about my future as an educator. But it’s really too early to tell, especially for secondary teaching. The only courses I’ve taught more than once are two graduate seminars in music informatics at Indiana University Bloomington. I taught one three times (and based it on a course I’d taught a couple of years earlier), the other only twice. I felt the last time for each course was no more successful than the first—that is to say, not very successful, despite the fact that I put a great deal of effort into revising them each time. However, in several respects, the situation was dramatically different from high-school math teaching. I was handicapped by the related facts that nothing was available I could use as a textbook, and that the state of the art was changing rapidly. And, of course, both the school and the course level were quite different from high school. So, while my difficulty improving lessons still seems like grounds for pessimism, I can’t take it too seriously.

Another factor is that I don’t have much experience dealing with student behavior issues: really, almost none since my first semester as a student teacher.

To summarize, I feel I’m *finally* prepared in terms of skills to be a successful high-school math teacher, but only in an environment where classroom management problems aren’t serious. But this remains to be seen. However, another factor is my age (65), health (very good), and energy level (very high for my age, but surely below average for new teachers). I recently wrote to the Woodrow Wilson Foundation to inform them that I find working as a regular full-time classroom teacher too stressful to be compatible with maintaining my health. Still, I’m open to and interested in teaching in other situations.

How I Plan To Teach Differently

1. To a great extent, students really are responsible for their own learning. Response: Make that as clear to students as I can; be available for a substantial amount of time every day and be sure they and (except for college classes) their parents know that; and keep track of who comes to get help and how long they stay. (Keeping track is important for dealing with parents and administrators as well as students.)

- 2. The appropriate preparation for a class varies greatly with students' age, maturity, etc. Response:** Prepare what I'll do in considerable detail for middle school classes, especially transitions. For high-school (or college) students, don't worry about transitions, but have written solutions to the more difficult homework problems. For honors classes, prepare as if for students a year (say) older.
- 3. When presenting content, the more concrete, the better. Response:** I'll try to be as concrete as possible at all times; do not assume it's not necessary without very good evidence! Use methods like the historical approach when appropriate. There's often a high-tech way to make things more concrete, e.g., with Geometers Sketchpad.
- 4. In classroom management, the more concrete, the better. Response:** Make my expectations as clear as possible, and make the consequences of violating them as clear as possible. If the consequences involve a period of time, use a program like "Don's Up/Down Timer".
- 5. It's important to have more than one way to explain anything. Response:** I'll devote at least some thought to a second way to explain almost everything; if possible, have any props—visual or other—handy that might be useful for the second explanation. Be ready to change course quickly based on students' understanding or lack of it.
- 6. Technology can help, both inside and outside class, but be careful with it. Response:** No change needed. I'll continue to think carefully about what I think students will get from the technology; start looking as early as possible for ways to give students hands-on use.
- 7. Some students' problems are beyond anything a classroom teacher can solve. Response:** I'll try to diagnose these problems as early as possible. Then, for a problem like "Isolde's", refer them to a counselor. For a problem like "Colton's", the school's Math Department or other remedial programs might be able to help; otherwise, refer them to a counselor.
- 8. It's important to be realistic about your strengths and weaknesses and adapt accordingly. An important special case: Grading homework and exams can easily overwhelm the teacher. Response:** Look for ways to use methods like those Levy suggested.
- 9. An element of discovery by students is tremendously valuable. Response:** I'll do my best to find and use activities that support student discovery, and to not help students too much. Plenty of material is available, e.g., examples in books like Polya (1957) and Lockhart (2009), articles in *Mathematics Teacher*, and resources from organizations like the Academy of Inquiry Based Learning (AIBL 2012), not to mention the ideas I've already collected in my "Wild and Crazy" list. Unfortunately, this item is something that—more than any of my others—is severely constrained by the administrations of a great many schools and school districts (and the administrations are arguably constrained by politics and public opinion).
- 10. One-tenth of a picture is worth 100 words. Response:** No change needed. For years, I've been on the alert for ways to use graphic communication principles in whatever I do; I'll continue to do so.
- 11. There are no lazy students, only unmotivated students. Response:** No change needed.

My Future As a Creator of Materials for Math Education

As my comments about class website above indicate, I've spent a fair amount of time these last few years creating documents for math education. Many of these are now available on the Woodrow Wilson Foundation's forum for teaching fellows as well as my own webpage.

See, for example, Infinite Bottles of Beer: Mathematical Concepts with Epsilon Pain (Byrd 2012b), Justin Price's Math Myths (Byrd 2012c), Meaningful Numbers and "Significant Figures" (Byrd 2012d), and Tips and Resources for Solving Math Problems (Byrd 2012e). Probably most important in the long run, I recently started a blog about math education, intended primarily for secondary-school math teachers, and I'm adding posts there that refer to many of these documents. The blog, "What I Learned About Why My Students Didn't Learn More", is at WhyMyStudentsDidnt.wordpress.com .

Acknowledgements

I owe a great deal to four groups of people: (1) IUPUI faculty I've had as instructors, site supervisors, and school liason; (2) my mentor teachers and other experienced teachers I've worked with; (3) the other Woodrow Wilson Fellows of the 2010 IUPUI cohort; and (4) the staff of the Woodrow Wilson Foundation, without whom none of this would have been possible. I'd like to single out, in group 1, Kathy Marrs, Crystal Hill Morton, Ofer Levy, and John Buckwalter; in group 2, Amber Krieger, Paul Weedling, John Miller, Sandra Warren, and Andrea Pendleton; and in group 4, Connie Bond, Dana DiLullo, and Audra Watson. But what of group 3, my fellow Fellows? Almost everyone really deserves to be singled out. So I'll just say: thanks so much to all for an extraordinarily rich experience, one I could not have imagined when I accepted the Fellowship.

Appendices

To save space, none of the appendices are included in this document. Those preceded by an asterisk (“*”) are available at

<http://www.informatics.indiana.edu/donbyrd/Teach/Math/>

The others may be available on request.

Appendix A. Mathematics Autobiography

Originally written for Secondary Mathematics Methods (N517); revised in March 2011.

Appendix B. Intellectual Autobiography

Originally written for Educational Psychology and Philosophy (P510); revised in April 2012. Some of its content also appears in Byrd (2010a).

Appendix C. Cultural Autobiography

Written for Diversity and Learning (S555) in July 2010.

***Appendix D. Zeno’s “Achilles and the Tortoise” Paradox vs. The Infinite Series**

Originally written for my interview for the Woodrow Wilson fellowship and revised several times.

Appendix E. My Philosophy of Secondary Teaching

Written for P510 in December 2010, based on my 2009 “Statement of Teaching” for college-level teaching, and revised in July 2011. NB: this is really more about techniques than philosophy.

***Appendix F. “Math is Cool, Fun, Wild, etc.” Teaching Ideas**

Appendix G. Similar Figures and Scale Lesson

A lesson for my middle-school honors students, December 2010.

Appendix H. Math is Wild

For my students at BCHS my last day there, in February 2012.

References

- AIBL (2012). The Academy of Inquiry Based Learning: IBL Resources. Retrieved April 30, 2012, from the World Wide Web: <http://www.inquirybasedlearning.org/?page=Resources>
- Barr, Avron (1982). Artificial Intelligence: Cognition as Computation. Defense Technical Information Center.
- Boaler, Jo (2008). *What’s Math Got to Do With It?* New York: Viking Penguin.
- Bressoud, David (2011a). Launchings, July 2011: The Worst Way to Teach. Retrieved April 20, 2012, from the World Wide Web: http://www.maa.org/columns/launchings/launchings_07_11.html

- Bressoud, David (2011b). Launchings, August 2011: The Best Way to Learn. Retrieved April 20, 2012, from the World Wide Web: <http://launchings.blogspot.com/2011/08/best-way-to-learn.html>
- Bruning, Roger, Gregory J. Schraw, Monica M. Norby, & Royce R. Ronning (2004). *Cognitive Psychology and Instruction*, 4th ed. Pearson.
- Byrd, Donald (2010a). Brief Autobiography. Retrieved April 10, 2012, from the World Wide Web: <http://www.informatics.indiana.edu/donbyrd/DonBiography.htm> . (Covers only through Spring 2010.)
- Byrd, Donald (2012a). Don's Up/Down Timer. Retrieved April 30, 2012, from the World Wide Web: <http://www.informatics.indiana.edu/donbyrd/Teach/Math/DABUpDownTimer>
- Byrd, Donald (2012b). Infinite Bottles Infinite Bottles of Beer: Mathematical Concepts with Epsilon Pain. Retrieved April 30, 2012, from the World Wide Web: http://www.informatics.indiana.edu/donbyrd/Teach/Math/InfiniteBottlesOfBeer_FullVer.pdf
- Byrd, Donald (2012c). Justin Price's Math Myths. Retrieved April 10, 2012, from the World Wide Web: http://www.informatics.indiana.edu/donbyrd/Teach/Math/Price_MathMythsDABRev.htm
- Byrd, Donald (2012d). Meaningful Numbers and "Significant Figures". Retrieved April 10, 2012, from the World Wide Web: <http://www.informatics.indiana.edu/donbyrd/Teach/Math/MeaningfulNumbers+SignificantFigures.pdf> . (Originally written for my calculus students at IUPUI, then revised.)
- Byrd, Donald (2012e). Tips and Resources for Solving Math Problems. Retrieved April 10, 2012, from the World Wide Web: <http://www.informatics.indiana.edu/donbyrd/Teach/Math/Tips+Resources4MathProblems.pdf> . (Originally written for my precalculus students my 2nd semester as a student teacher and revised numerous times for subsequent classes.)
- CMI (2012). The History and Promise of IBL in Math Education. Center for Mathematical Inquiry, UCSB Department of Mathematics. Retrieved April 10, 2012, from the World Wide Web: http://math.ucsb.edu/department/cmi/IBL_History.html
- Driscoll, Marcy (2005). *Psychology of Learning for Instruction*, 3rd ed. Pearson.
- Flanagan, Owen (1991). *The Science of the Mind*. MIT Press.
- Grinstead, Charles, & Snell, J. Laurie (2007). *Introduction to Probability*. American Mathematical Society.
- Halmos, P. R., E. E. Moise, and G. Piranian (1975, May). The Problem of Learning to Teach. *American Mathematical Monthly* 82, no. 5, pp. 466–476.
- Hiebert, James (2003). What Research Says about the NCTM Standards. In *A Research Companion to "Principles and Standards for School Mathematics"*, ed. by Jeremy Kilpatrick, W. Gary Martin, and Deborah Schifter, pp. 5–23. Reston, Va.: National Council of Teachers of Mathematics.
- Hofstadter, Douglas (1985). Waking up from the Boolean Dream, or, Subcognition as Computation (Chapter 26), in *Metamagical Themas*. New York: Basic Books.
- Hofstadter, Douglas (2007). *I Am a Strange Loop*. New York: Basic Books.
- Holt, John (1982). *How Children Fail*, revised ed. Addison-Wesley.

- Indiana Dept. of Education (2006a). *2000 Mathematics Standards: Grade 6*. Retrieved April 5, 2012, from the World Wide Web:
<http://dc.doe.in.gov/Standards/AcademicStandards/PrintLibrary/math.shtml>
- Indiana Dept. of Education (2006b). *2000 Mathematics Standards: Grade 7*. Retrieved April 5, 2012, from the World Wide Web:
<http://dc.doe.in.gov/Standards/AcademicStandards/PrintLibrary/math.shtml>
- Indiana Dept. of Education (2006c). *2000 Mathematics Standards: Algebra II*. Retrieved April 5, 2012, from the World Wide Web:
<http://dc.doe.in.gov/Standards/AcademicStandards/PrintLibrary/math.shtml>
- Kaser, Catherine H. (2007). Series on Highly Effective Practices—6. Successfully Managing Student Transitions. Retrieved April 30, 2012, from the World Wide Web:
<http://education.odu.edu/esse/research/series/transitions.shtml>
- Katz, Victor J. (2009). *A History of Mathematics: An Introduction*, 3rd ed. Addison-Wesley.
- Lockhart, Paul (2009). *A Mathematician's Lament*. New York: Bellevue Literary Press. Part I of the book (the "Lament" proper) retrieved May 4, 2012, from the World Wide Web:
www.maa.org/devlin/lockhartslament.pdf
- Mazer, Jesse (2011). Talk:Albert Einstein [talk page for discussing improvements to the Albert Einstein article]. Retrieved April 30, 2012, from the World Wide Web:
http://en.wikiquote.org/wiki/Talk:Albert_Einstein
- Middleton, James A. and Jansen, Amanda (2011). *Motivation Matters and Interest Counts: Fostering Engagement in Mathematics*. Reston, Va.: National Council of Teachers of Mathematics.
- Minsky, Marvin (1988). *The Society of Mind*. New York: Simon and Schuster.
- Nabb, Keith (2010, December/January). A Close Encounter with Infinity: Inventing New Mathematics. *Mathematics Teacher* 104, 5, pp. 373–378.
- National Research Council (2000). *How People Learn: Brain, Mind, Experience, and School*, expanded ed. National Academy Press.
- Phillips, D. C., and Jonas F. Solits (2004). *Perspectives on Learning*. New York: College Press.
- Polya, George (1957). *How to Solve It*, 2nd ed. Princeton University Press.
- Richeson, A. W. (1935). Warren Colburn and His Influence on Arithmetic in the United States. *National Mathematics Magazine* 10, no. 3, Dec. 1935, pp. 73–79.
- Sawyer, W.W. (1982). *Prelude to Mathematics*. New York: Dover Publications.
- vos Savant, Marilyn (2006). Game Show Problem. Retrieved September 26, 2012, from the World Wide Web: <http://marilynvosavant.com/game-show-problem/>.
- vos Savant, Marilyn (1990a, September 9). Ask Marilyn. *Parade Magazine*, p. 15.
- vos Savant, Marilyn (1990b, December 2). Ask Marilyn. *Parade Magazine*, p. 25.
- vos Savant, Marilyn (1991, February 17). Ask Marilyn. *Parade Magazine*, p. 12.