A Computational Interpretation of Classical S4 Modal Logic

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Plans are programs, and programs are proofs. In particular, multiagent plans are distributed programs, and distributed programs are modal proofs. Inspired by these slogans, I present a new proof system for classical S4 modal logic, based most directly on Wadler's system for classical propositional logic (2003) and Ghani, de Paiva, and Ritter's system for intuitionistic modal logic (1998). The system generalizes to multiple S4-modalities and implications among them, thus modeling multiple agents that share references to proof terms and perform distributed computations by confluent reductions.

I. INTRODUCTION

"Can you both make it on Tuesday at noon?", said Alice to Bob and Carol, trying to schedule a joint meeting among the three of them. Her question expresses the shared plan below.

- Bob knows if he is available, and announces it.
- Carol knows if she is available, and announces it.
- The logical AND of the two boolean values can then be computed.

One way to formalize this plan is the proof tree below, in which a formula like $\mathsf{A}\,\Box X$ means that Alice knows a boolean value, and a sequent like $\mathsf{B}\,\Box X, \mathsf{C}\,\Box X \to \mathsf{A}\,\Box X$ means that, if Bob and Carol each know a boolean value, then so can Alice.

(1)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{A}\,\Box X, \mathsf{A}\,\Box X \to X}\,\Box\mathrm{IdR} \qquad \dfrac{}{\mathsf{A}\,\Box X, \mathsf{A}\,\Box X \to X}\,\Box\mathrm{IdR}}{\mathsf{A}\,\Box X, \mathsf{A}\,\Box X \to X}\,\mathrm{And}}{\mathsf{A}\,\Box X, \mathsf{A}\,\Box X \to \mathsf{A}\,\Box X}\,\Box\mathrm{R}}{\mathsf{B}\,\Box X, \mathsf{A}\,\Box X \to \mathsf{A}\,\Box X}\,\mathsf{A/B}\,\Box\mathrm{LI}}{\mathsf{B}\,\Box X, \mathsf{C}\,\Box X \to \mathsf{A}\,\Box X}\,\mathsf{A/C}\,\Box\mathrm{LI}$$

This proof tree, read from top to bottom, can be interpreted as follows.

- The $\Box$IdR rule says that, if Alice knows a boolean, then there is one. We apply this rule twice to deduce that, if Alice knows two booleans, then there are two.
- The And rule says that, for any two booleans, there is a third boolean that is their logical AND.
- Consequently, the $\Box$R rule says that, if Alice knows two booleans, then she can compute a third boolean that is their logical AND.
- Since Bob and Carol can both speak to Alice, we then apply two $\Box$LI rules in succession, concluding that, given that Bob knows a boolean and Carol knows a boolean, Alice can know a boolean as well.

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The goal of this paper is to turn modal proofs like (1) into a distributed program that agents like Alice, Bob, and Carol can together execute as a plan. This transformation of proofs into programs takes two steps. First, we decorate each proof with a program term; for example, as shown in Figure 1, the proof (1) can be decorated with the program term

(2) $$[\![c]\!]_{\mathsf{C}} \bullet [\![c_0]\!]^{\mathsf{A}}_{\mathsf{C}}.([\![b]\!]_{\mathsf{B}} \bullet [\![b_0]\!]^{\mathsf{A}}_{\mathsf{B}}.([\![b_0 \text{ AND } c_0]\!]_{\mathsf{A}} \bullet \bar{x})),$$

in which

- b denotes Bob's knowledge of his own availability, shown as $\mathsf{B}\; b : X$ in the figure;
- c denotes Carol's knowledge of her own availability, shown as $\mathsf{C}\; c : X$ in the figure; and
- $\bar{x}$ denotes the continuation once the plan is completed, shown as $\bar{x} : \mathsf{A}\,\Box X$ in the figure.

Roughly speaking, the Cut operator $\bullet$ combines two terms by function application, and the body of a function abstraction is enclosed by a pair of parentheses. Double brackets, such as those surrounding $[\![b_0 \text{ AND } c_0]\!]_{\mathsf{A}}$, mean to enter a new modal proof context, and correspond to the $\Box$R deduction rule.

Second, we specify term reductions that model program execution. Suppose that Bob answers "no" to Alice's question. Without waiting for Carol's response, Alice would be able to deduce that it is not the case that Bob and Carol can both make it on Tuesday at noon. To model this deduction, we substitute FALSE for b in (2), then perform the following sequence of reductions.

(3)
$$\begin{aligned}
&[\![c]\!]_{\mathsf{C}} \bullet [\![c_0]\!]^{\mathsf{A}}_{\mathsf{C}}.([\![\text{FALSE}]\!]_{\mathsf{B}} \bullet [\![b_0]\!]^{\mathsf{A}}_{\mathsf{B}}.([\![b_0 \text{ AND } c_0]\!]_{\mathsf{A}} \bullet \bar{x}))\\
&\quad\Rightarrow [\![c]\!]_{\mathsf{C}} \bullet [\![c_0]\!]^{\mathsf{A}}_{\mathsf{C}}.([\![\text{FALSE AND } c_0]\!]_{\mathsf{A}} \bullet \bar{x})\\
&\quad\Rightarrow [\![c]\!]_{\mathsf{C}} \bullet [\![c_0]\!]^{\mathsf{A}}_{\mathsf{C}}.([\![\text{FALSE}]\!]_{\mathsf{A}} \bullet \bar{x})\\
&\quad\Rightarrow [\![\text{FALSE}]\!]_{\mathsf{A}} \bullet \bar{x}
\end{aligned}$$

Our reduction rules model distributed computation, because they are amenable to implementation in the form of an algorithm whose execution is distributed among multiple agents that communicate with each other.

II. BACKGROUND

The idea of turning proof trees into program terms, and proof reduction into program execution, is the celebrated

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{A}\,b_0 : X,\ \mathsf{A}\,c_0 : X \rhd \cdot \mid b_0 : X}\,\square\mathrm{IdR} \qquad \dfrac{}{\mathsf{A}\,b_0 : X,\ \mathsf{A}\,c_0 : X \rhd \cdot \mid c_0 : X}\,\square\mathrm{IdR}}{\mathsf{A}\,b_0 : X,\ \mathsf{A}\,c_0 : X \rhd \cdot \mid b_0\ \textsc{and}\ c_0 : X}\,\mathrm{And}}{\mathsf{A}\,b_0 : X,\ \mathsf{A}\,c_0 : X,\ \mathsf{B}\,b : X,\ \mathsf{C}\,c : X \rhd \bar{x} : \mathsf{A}\,\square X \mid \llbracket b_0\ \textsc{and}\ c_0 \rrbracket_{\mathsf{A}} : \mathsf{A}\,\square X}\,\square\mathrm{R}}{\mathsf{A}\,b_0 : X,\ \mathsf{A}\,c_0 : X,\ \mathsf{B}\,b : X,\ \mathsf{C}\,c : X \rhd \llbracket b_0\ \textsc{and}\ c_0 \rrbracket_{\mathsf{A}} \bullet \bar{x} \lhd \bar{x} : \mathsf{A}\,\square X}\,\mathrm{RE}}{\llbracket b_0 \rrbracket^{\mathsf{A}}_{\mathsf{B}}.(\llbracket b_0\ \textsc{and}\ c_0 \rrbracket_{\mathsf{A}} \bullet \bar{x}) : \mathsf{B}\,\square X \mid \mathsf{A}\,c_0 : X,\ \mathsf{B}\,b : X,\ \mathsf{C}\,c : X \lhd \bar{x} : \mathsf{A}\,\square X}\,\mathsf{A/B}\,\square\mathrm{LI}}{\mathsf{A}\,c_0 : X,\ \mathsf{B}\,b : X,\ \mathsf{C}\,c : X \rhd \llbracket b \rrbracket_{\mathsf{B}} \bullet \llbracket b_0 \rrbracket^{\mathsf{A}}_{\mathsf{B}}.(\llbracket b_0\ \textsc{and}\ c_0 \rrbracket_{\mathsf{A}} \bullet \bar{x}) \lhd \bar{x} : \mathsf{A}\,\square X}\,\square\square\mathrm{LE}}{\llbracket c_0 \rrbracket^{\mathsf{A}}_{\mathsf{C}}.(\llbracket b \rrbracket_{\mathsf{B}} \bullet \llbracket b_0 \rrbracket^{\mathsf{A}}_{\mathsf{B}}.(\llbracket b_0\ \textsc{and}\ c_0 \rrbracket_{\mathsf{A}} \bullet \bar{x})) : \mathsf{C}\,\square X \mid \mathsf{B}\,b : X,\ \mathsf{C}\,c : X \lhd \bar{x} : \mathsf{A}\,\square X}\,\mathsf{A/C}\,\square\mathrm{LI}}{\mathsf{B}\,b : X,\ \mathsf{C}\,c : X \rhd \llbracket c \rrbracket_{\mathsf{C}} \bullet \llbracket c_0 \rrbracket^{\mathsf{A}}_{\mathsf{C}}.(\llbracket b \rrbracket_{\mathsf{B}} \bullet \llbracket b_0 \rrbracket^{\mathsf{A}}_{\mathsf{B}}.(\llbracket b_0\ \textsc{and}\ c_0 \rrbracket_{\mathsf{A}} \bullet \bar{x})) \lhd \bar{x} : \mathsf{A}\,\square X}\,\square\square\mathrm{LE}
$$

Figure 1: Decorating each node in the proof tree (1) with a program term

Curry-Howard correspondence. This correspondence treats propositions in a logic as types in a programming language, and proofs of these propositions as programs of these types. For example, Church's (1932, 1940) λ-calculus is not just a functional programming language but also a proof system for intuitionistic propositional logic: The expression

(4) $$3 \times 4 + 5$$

is not just a program that computes an integer but also a constructive proof that there exists an integer. If I denotes the proposition that there exists an integer, then this proof uses three axioms—called 3, 4, and 5—that conclude I, and two further axioms—called $\times$ and $+$—that conclude $I \supset I \supset I$. We can also read the expression (4) as a *plan* for computing an integer; as such, the plan makes certain assumptions about the arithmetic capabilities of the executing agent, named by the axioms 3, 4, 5, $\times$, and $+$.

The Curry-Howard correspondence is relevant to discourse processing and multiagent planning, because plans can be viewed as programs (Stone 2005). On one hand, if plans are programs, then multiagent plans are distributed programs. On the other hand, if programs are proofs, then distributed programs are proofs of knowledge and action. One example was given at the beginning of this paper. Another example is the following dialog.[1] In this dialog, Alice and Bob construct a shared plan to send a letter.

> Alice
>
> When you have time, write a letter to Carol and thank her a bit.
>
> Bob
>
> In your next letter then, could you please give me her address? I don't know if I have it here.
>
> Alice
>
> Okay.

[1] This fictional dialog is based on a Mandarin Chinese telephone conversation in the CallHome corpus (part of `ma_0003.txt`).

Because Alice and Bob can reason about knowledge states and preconditions in the future, their plan makes sense even though Alice never gives Carol's address to Bob during this conversation.

To study multiagent plans, then, I develop here the Curry-Howard correspondence for one particular logic of knowledge and action, namely classical S4 modal logic. In other words, I formulate classical S4 so as to treat propositions as types in a distributed programming language, and proofs of these propositions as distributed programs of these types. The system straightforwardly generalizes to multiple S4-modalities and implications between them.

Though the Curry-Howard correspondence has been studied extensively for intuitionistic S4 (Bierman and de Paiva 2000; Davies and Pfenning 1996; Goubault-Larrecq 1996; Martini and Masini 1996; inter alia), it has not been considered explicitly for classical S4 in the literature. As Matthew Stone noted in personal communication, the law of the excluded middle is valid in the real world: Bob is either free or busy at any given time. Thus plans of knowledge and action in the real world should be viewed as classical proofs, not intuitionistic ones, and it is for the former that the Curry-Howard correspondence is developed here.

III. A Sequent Calculus for Classical S4

I start with Wadler's (2003) *dual calculus*, a crisp presentation of the Curry-Howard correspondence for classical propositional (non-modal) logic, and extend it with a dual pair of S4-modalities.

Figure 2 shows the syntax and inference rules of a sequent calculus for classical S4 modal logic. This calculus extends Gentzen's sequent calculus for classical propositional logic, as presented by Wadler (2003), with a pair of dual modalities $\square$ and $\diamond$. We write A, B for formulas, each of which can be an atomic formula X, a conjunction $A \mathbin{\&} B$, a disjunction $A \vee B$, a negation $\neg A$, a necessity $\square A$, or a possibility $\diamond A$. Logical implication $A \supset B$ can be defined as syntactic sugar for $\neg A \vee B$ (in the call-by-name case) or $\neg(A \mathbin{\&} \neg B)$ (in the call-by-value case).

Formula $A, B ::= X \mid A \,\&\, B \mid A \vee B \mid \Box A \mid \Diamond A \mid \neg A$

Antecedent $\Gamma ::= A_1, \ldots, A_m$

Succeedent $\Theta ::= B_1, \ldots, B_m$

Sequent $\Gamma \to \Theta$

$$\frac{}{A, \Gamma \to \Theta, A}\,\mathrm{Id}$$

$$\frac{}{\Gamma, \Box A \to \Theta, A}\,\Box\mathrm{IdR} \qquad \frac{}{A, \Gamma \to \Diamond A, \Theta}\,\Diamond\mathrm{IdL}$$

$$\frac{\Box\Delta \to \Diamond\Phi, A}{\Gamma, \Box\Delta \to \Diamond\Phi, \Theta, \Box A}\,\Box\mathrm{R} \qquad \frac{A, \Box\Delta \to \Diamond\Phi}{\Diamond A, \Gamma, \Box\Delta \to \Diamond\Phi, \Theta}\,\Diamond\mathrm{L}$$

$$\frac{\Gamma \to \Theta, A \quad \Gamma \to \Theta, B}{\Gamma \to \Theta, A \,\&\, B}\,\&\mathrm{R} \qquad \frac{A, \Gamma \to \Theta}{A \,\&\, B, \Gamma \to \Theta} \quad \frac{B, \Gamma \to \Theta}{A \,\&\, B, \Gamma \to \Theta}\,\&\mathrm{L}$$

$$\frac{\Gamma \to \Theta, A}{\Gamma \to \Theta, A \vee B} \quad \frac{\Gamma \to \Theta, B}{\Gamma \to \Theta, A \vee B}\,\vee\mathrm{R} \qquad \frac{A, \Gamma \to \Theta \quad B, \Gamma \to \Theta}{A \vee B, \Gamma \to \Theta}\,\vee\mathrm{L}$$

$$\frac{A, \Gamma \to \Theta}{\Gamma \to \Theta, \neg A}\,\neg\mathrm{R} \qquad \frac{\Gamma \to \Theta, A}{\neg A, \Gamma \to \Theta}\,\neg\mathrm{L}$$

$$\frac{\Gamma \to \Theta, A \quad A, \Gamma \to \Theta}{\Gamma \to \Theta}\,\mathrm{Cut}$$

Figure 2: Gentzen's sequent calculus, extended with dual modalities $\Box$ and $\Diamond$

Under the Curry-Howard correspondence, formulas are equivalent to types:

- Atomic formulas are primitive types.
- Conjunctions are product (record) types.
- Disjunctions are sum (tagged union) types.
- Negations are continuation types.
- Necessities are quotation types, in some sense that the present work explicates formally.
- Possibilities are, dually, continuation quotation types.
- Logical implications are function types.

Sequents are of the form

(5) $$A_1, \ldots, A_m \to B_1, \ldots, B_n,$$

or simply $\Gamma \to \Theta$, where the antecedent Γ and the succedent Θ are each an unordered bag of formulas, of assumptions and conclusions, respectively. The intended interpretation of a sequent is that the conjunction of the antecedent formulas to the left entail the disjunction of the succedent formulas to the right; in other words, if every assumption holds, then some conclusion holds.

It may help intuition to note that the usual structural rules of Exchange, Thinning, and Contraction are admissible:

- Because the formulas on each side of a sequent are an unordered bag, Exchange is admissible.
- Because the Id rule allows for extra formulas on either side of the sequent, Thinning is admissible.
- Because assumptions and conclusions are duplicated in the &R, $\vee$L, and Cut rules, Contraction is admissible.

To support the modalities $\Box$ and $\Diamond$, we extend Gentzen's system with four new deduction rules.

- The $\Box$R and $\Diamond$L rules are modality introduction rules. These two rules embody the 4 axiom (frame transitivity) for S4. They are identical to the νR and πL rules in Fitting's (1983) sequent calculus for S4 (page 90).

- The $\Box$IdR and $\Diamond$IdL rules deduce A from $\Box A$ and $\Diamond A$ from A. These two rules implement the T axiom (frame reflexivity) for S4. They are analogous, but not identical, to Fitting's νL and πR rules for modality elimination. Indeed, Figure 3 shows how to derive the latter rules (renamed $\Box$L and $\Diamond$R for consistency) from $\Box$IdR and $\Diamond$IdL, together with Cut.

One of the most pleasing aspects of Gentzen's sequent calculus is that it is dual to itself under the mapping that switches

$$\dfrac{A, \Gamma, \Box A \rightarrow \Theta}{\Gamma, \Box A \rightarrow \Theta}\,\Box\mathrm{L} \qquad \equiv \qquad \dfrac{\dfrac{}{\Gamma, \Box A \rightarrow \Theta, A}\,\Box\mathrm{IdR} \qquad A, \Gamma, \Box A \rightarrow \Theta}{\Gamma, \Box A \rightarrow \Theta}\,\mathrm{Cut}$$

$$\dfrac{\Gamma \rightarrow \Diamond A, \Theta, A}{\Gamma \rightarrow \Diamond A, \Theta}\,\Diamond\mathrm{R} \qquad \equiv \qquad \dfrac{\Gamma \rightarrow \Diamond A, \Theta, A \qquad \dfrac{}{A, \Gamma \rightarrow \Diamond A, \Theta}\,\Diamond\mathrm{IdL}}{\Gamma \rightarrow \Diamond A, \Theta}\,\mathrm{Cut}$$

Figure 3: Derived rules for modality elimination

conjunction with disjunction, and antecedent with succedent, while leaving negation intact. This duality, explored at length in Wadler's paper, is extended here to one between $\Box$ and $\Diamond$.

IV. THE MODAL DUAL CALCULUS

The modal dual calculus, shown in Figure 4, decorates the S4 sequent calculus presented above with proof terms. This term calculus extends Wadler's system by dividing each side of a sequent into two *zones*: An antecedent in a sequent is now of the form

(6) $$A_1, \ldots, A_m \,;\, B_1, \ldots, B_n,$$

or simply $\Gamma \,;\, \Delta$, where the formulas before the semicolon constitute the *classical zone*, and the formulas after the semicolon constitute the *modal zone*. Dually, an succedent in a sequent is now of the form

(7) $$C_1, \ldots, C_k \,;\, D_1, \ldots, D_l,$$

or simply $\Phi \,;\, \Theta$, where—mirroring horizontally—the formulas before the semicolon constitute the modal zone, and the formulas after the semicolon constitute the classical zone. Note that, just like the comma, the semicolon is part of the structural punctuation of the antecedent rather than a logical connective.

The intended interpretation of the modal zones is for the modal assumptions $B_1, \ldots, B_n$ to be implicitly boxed, and the modal conclusions $C_1, \ldots, C_k$ to be implicitly diamonded. By structurally distinguishing modal formulas from classical ones in the environment of a sequent, this system achieves such desirable proof-theoretic properties as a modal substitution lemma. This *dual-zone* approach to modal proofs has been successfully applied to other modal and linear logics (Barber 1996; Ghani et al. 1998; Girard 1993; Schellinx 1996; Wadler 1993, 1994); the calculus here is most directly based on Ghani et al.'s (1998) Dual Intuitionistic Modal Logic.

As is usual for the proofs-as-programs approach, each assumption in a sequent is labeled with a *variable*. Less usual is the use, inherited from Wadler's dual calculus, of *covariables* to label each conclusion in a sequent. Different (co)variables are used to label formulas in each zone:

- classical variables x, y, z;
- modal variables a, b, c;
- modal covariables $\bar{a}, \bar{b}, \bar{c}$; and
- classical covariables $\bar{x}, \bar{y}, \bar{z}$.

A different Id deduction rule is used for (co)variables of each sort: respectively IdR, $\Box$IdR, $\Diamond$IdL, and IdL. Using these Id rules and the Cut rule, many useful structural rules can be derived in this calculus, including those shown in Figure 5. Two of those rules, RE and $\Box\Box$LE, are used in the opening example in Figure 1 on page 2.

Decorating the modality introduction rules $\Box$R and $\Diamond$L are the term constructor $\llbracket\ \rrbracket$ and the coterm constructor $\langle\!|\ |\!\rangle$, respectively.

Just as with Wadler's dual calculus, the modal dual calculus would not be confluent if it were equipped with the general Cut elimination reduction rule that is obvious. Two derivatives of the calculus exist that are confluent: a call-by-value version and a call-by-name version. Typing and reduction rules for the call-by-value modal dual calculus are presented in Figures 6 and 8; those for the call-by-name version are presented in Figures 7 and 9.

To achieve confluence and extensional $\beta\eta$-equality, classical assumptions in the call-by-value calculus are restricted to atomic and possibility ($\Diamond$) types only. Meanwhile, the recursive definition of coterms is modified to use a single syntactic form for coabstraction: an unordered bag of patterns v_i paired with statements S_i, notated

(8) $$v_1.(S_1) \;\Box\; \cdots \;\Box\; v_m.(S_m).$$

By definition (see the top part of Figure 6), all patterns v_i are in η-long normal form; for instance, the only valid pattern (up to α-conversion) for the type $X \,\&\, Y$, where X and Y are atomic, is $\langle x, y \rangle$. A (well-typed) coterm of the above form, when cut against a value term V of the same type, is guaranteed to contain exactly one branch $v_i.(S_i)$ whose pattern part v_i matches V. Depending on the (type of the) pattern, the coabstraction may bind zero or more classical assumptions, modal assumptions, and classical conclusions simultaneously. Thus, as shown in parts of Figure 6, the typing rules &L, $\vee$L, $\neg$L, and $\Box$LI, which are coterm constructors, are derived rather than primitive in the call-by-value version of the calculus.

As for the call-by-name version of the modal dual calculus, confluence and extensional $\beta\eta$-equality are achieved dually. Classical conclusions are restricted to atomic and necessity ($\Box$) types only. Meanwhile, the recursive definition of terms is modified to use a single syntactic form for abstraction: an

Type	$A, B ::= X \mid A \mathbin{\&} B \mid A \vee B \mid \Box A \mid \Diamond A \mid \neg A$
Term	$M, N ::= x \mid a \mid \llbracket M \rrbracket \mid \langle M, N \rangle \mid \langle M \rangle\mathrm{inl} \mid \langle N \rangle\mathrm{inr} \mid [K]\mathrm{not} \mid (S).\bar{x} \mid (S).\langle\!\lbrack \bar{a} \rbrack\!\rangle$
Coterm	$K, L ::= \bar{x} \mid \bar{a} \mid \langle\!\lbrack K \rbrack\!\rangle \mid [K, L] \mid \mathrm{fst}[K] \mid \mathrm{snd}[L] \mid \mathrm{not}\langle M \rangle \mid x.(S) \mid \llbracket a \rrbracket.(S)$
Statement	$S ::= M \bullet K$
Classical antecedent	$\Gamma ::= x_1 : A_1, \ldots, x_m : A_m$
Modal antecedent	$\Delta ::= a_1 : A_1, \ldots, a_m : A_m$
Modal succeedent	$\Phi ::= \bar{b}_1 : B_1, \ldots, \bar{b}_m : B_m$
Classical succeedent	$\Theta ::= \bar{y}_1 : B_1, \ldots, \bar{y}_m : B_m$
Right sequent	$\Gamma ; \Delta \rhd \Phi ; \Theta \mid M : A$
Left sequent	$K : A \mid \Gamma ; \Delta \lhd \Phi ; \Theta$
Center sequent	$\Gamma ; \Delta \rhd S \lhd \Phi ; \Theta$

$$\frac{}{x : A, \Gamma ; \Delta \rhd \Phi ; \Theta \mid x : A}\,\mathrm{IdR} \qquad \frac{}{\bar{x} : A \mid \Gamma ; \Delta \lhd \Phi ; \Theta, \bar{x} : A}\,\mathrm{IdL}$$

$$\frac{}{\Gamma ; \Delta, a : A \rhd \Phi ; \Theta \mid a : A}\,\Box\mathrm{IdR} \qquad \frac{}{\bar{a} : A \mid \Gamma ; \Delta \lhd \bar{a} : A, \Phi ; \Theta}\,\Diamond\mathrm{IdL}$$

$$\frac{\cdot ; \Delta \rhd \Phi ; \cdot \mid M : A}{\Gamma ; \Delta \rhd \Phi ; \Theta \mid \llbracket M \rrbracket : \Box A}\,\Box\mathrm{R} \qquad \frac{K : A \mid \cdot ; \Delta \lhd \Phi ; \cdot}{\langle\!\lbrack K \rbrack\!\rangle : \Diamond A \mid \Gamma ; \Delta \lhd \Phi ; \Theta}\,\Diamond\mathrm{L}$$

$$\frac{\Gamma ; \Delta \rhd \Phi ; \Theta \mid M : A \quad \Gamma ; \Delta \rhd \Phi ; \Theta \mid N : B}{\Gamma ; \Delta \rhd \Phi ; \Theta \mid \langle M, N \rangle : A \mathbin{\&} B}\,\&\mathrm{R} \qquad \frac{K : A \mid \Gamma ; \Delta \lhd \Phi ; \Theta}{\mathrm{fst}[K] : A \mathbin{\&} B \mid \Gamma ; \Delta \lhd \Phi ; \Theta} \quad \frac{L : B \mid \Gamma ; \Delta \lhd \Phi ; \Theta}{\mathrm{snd}[L] : A \mathbin{\&} B \mid \Gamma ; \Delta \lhd \Phi ; \Theta}\,\&\mathrm{L}$$

$$\frac{\Gamma ; \Delta \rhd \Phi ; \Theta \mid M : A}{\Gamma ; \Delta \rhd \Phi ; \Theta \mid \langle M \rangle\mathrm{inl} : A \vee B} \quad \frac{\Gamma ; \Delta \rhd \Phi ; \Theta \mid N : B}{\Gamma ; \Delta \rhd \Phi ; \Theta \mid \langle N \rangle\mathrm{inr} : A \vee B}\,\vee\mathrm{R} \qquad \frac{K : A \mid \Gamma ; \Delta \lhd \Phi ; \Theta \quad L : B \mid \Gamma ; \Delta \lhd \Phi ; \Theta}{[K, L] : A \vee B \mid \Gamma ; \Delta \lhd \Phi ; \Theta}\,\vee\mathrm{L}$$

$$\frac{K : A \mid \Gamma ; \Delta \lhd \Phi ; \Theta}{\Gamma ; \Delta \rhd \Phi ; \Theta \mid [K]\mathrm{not} : \neg A}\,\neg\mathrm{R} \qquad \frac{\Gamma ; \Delta \rhd \Phi ; \Theta \mid M : A}{\mathrm{not}\langle M \rangle : \neg A \mid \Gamma ; \Delta \lhd \Phi ; \Theta}\,\neg\mathrm{L}$$

$$\frac{\Gamma ; \Delta \rhd S \lhd \Phi ; \Theta, \bar{x} : A}{\Gamma ; \Delta \rhd \Phi ; \Theta \mid (S).\bar{x} : A}\,\mathrm{RI} \qquad \frac{x : A, \Gamma ; \Delta \rhd S \lhd \Phi ; \Theta}{x.(S) : A \mid \Gamma ; \Delta \lhd \Phi ; \Theta}\,\mathrm{LI}$$

$$\frac{\Gamma ; \Delta \rhd S \lhd \bar{a} : A, \Phi ; \Theta}{\Gamma ; \Delta \rhd \Phi ; \Theta \mid (S).\langle\!\lbrack \bar{a} \rbrack\!\rangle : \Diamond A}\,\Diamond\mathrm{RI} \qquad \frac{\Gamma ; \Delta, a : A \rhd S \lhd \Phi ; \Theta}{\llbracket a \rrbracket.(S) : \Box A \mid \Gamma ; \Delta \lhd \Phi ; \Theta}\,\Box\mathrm{LI}$$

$$\frac{\Gamma ; \Delta \rhd \Phi ; \Theta \mid M : A \quad K : A \mid \Gamma ; \Delta \lhd \Phi ; \Theta}{\Gamma ; \Delta \rhd M \bullet K \lhd \Phi ; \Theta}\,\mathrm{Cut}$$

Figure 4: Wadler's dual calculus, extended with dual modalities $\Box$ and $\Diamond$

$$\frac{}{x:A,\Gamma;\Delta \rhd x \bullet \bar{x} \lhd \Phi;\Theta,\bar{x}:A}\,\mathrm{Id} \quad\equiv\quad \frac{\frac{}{x:A,\Gamma;\Delta\rhd\Phi;\Theta,\bar{x}:A \mid x:A}\,\mathrm{IdR} \qquad \frac{}{\bar{x}:A\mid x:A,\Gamma;\Delta\lhd\Phi;\Theta,\bar{x}:A}\,\mathrm{IdL}}{x:A,\Gamma;\Delta\rhd x\bullet\bar{x}\lhd\Phi;\Theta,\bar{x}:A}\,\mathrm{Cut}$$

$$\frac{}{\Gamma;\Delta,a:A \rhd a \bullet \bar{x} \lhd \Phi;\Theta,\bar{x}:A}\,\Box\mathrm{Id} \quad\equiv\quad \frac{\frac{}{\Gamma;\Delta,a:A\rhd\Phi;\Theta,\bar{x}:A \mid a:A}\,\Box\mathrm{IdR} \qquad \frac{}{\bar{x}:A\mid \Gamma;\Delta,a:A\lhd\Phi;\Theta,\bar{x}:A}\,\mathrm{IdL}}{\Gamma;\Delta,a:A\rhd a\bullet\bar{x}\lhd\Phi;\Theta,\bar{x}:A}\,\mathrm{Cut}$$

$$\frac{}{x:A,\Gamma;\Delta \rhd x \bullet \bar{a} \lhd \bar{a}:A,\Phi;\Theta}\,\Diamond\mathrm{Id} \quad\equiv\quad \frac{\frac{}{x:A,\Gamma;\Delta\rhd\bar{a}:A,\Phi;\Theta \mid x:A}\,\mathrm{IdR} \qquad \frac{}{\bar{a}:A\mid x:A,\Gamma;\Delta\lhd\bar{a}:A,\Phi;\Theta}\,\Diamond\mathrm{IdL}}{x:A,\Gamma;\Delta\rhd x\bullet\bar{a}\lhd\bar{a}:A,\Phi;\Theta}\,\mathrm{Cut}$$

$$\frac{}{\Gamma;\Delta,a:A \rhd a \bullet \bar{a} \lhd \bar{a}:A,\Phi;\Theta}\,\Box\Diamond\mathrm{Id} \quad\equiv\quad \frac{\frac{}{\Gamma;\Delta,a:A\rhd\bar{a}:A,\Phi;\Theta \mid a:A}\,\Box\mathrm{IdR} \qquad \frac{}{\bar{a}:A\mid \Gamma;\Delta,a:A\lhd\bar{a}:A,\Phi;\Theta}\,\Diamond\mathrm{IdL}}{\Gamma;\Delta,a:A\rhd a\bullet\bar{a}\lhd\bar{a}:A,\Phi;\Theta}\,\mathrm{Cut}$$

$$\frac{\Gamma;\Delta\rhd\Phi;\Theta,\bar{x}:A \mid M:A}{\Gamma;\Delta \rhd M \bullet \bar{x} \lhd \Phi;\Theta,\bar{x}:A}\,\mathrm{RE} \quad\equiv\quad \frac{\Gamma;\Delta\rhd\Phi;\Theta,\bar{x}:A \mid M:A \qquad \frac{}{\bar{x}:A\mid \Gamma;\Delta\lhd\Phi;\Theta,\bar{x}:A}\,\mathrm{IdL}}{\Gamma;\Delta\rhd M\bullet\bar{x}\lhd\Phi;\Theta,\bar{x}:A}\,\mathrm{Cut}$$

$$\frac{\Gamma;\Delta\rhd\bar{a}:A,\Phi;\Theta \mid M:A}{\Gamma;\Delta \rhd M \bullet \bar{a} \lhd \bar{a}:A,\Phi;\Theta}\,\Diamond\mathrm{RE} \quad\equiv\quad \frac{\Gamma;\Delta\rhd\bar{a}:A,\Phi;\Theta \mid M:A \qquad \frac{}{\bar{a}:A\mid \Gamma;\Delta\lhd\bar{a}:A,\Phi;\Theta}\,\Diamond\mathrm{IdL}}{\Gamma;\Delta\rhd M\bullet\bar{a}\lhd\bar{a}:A,\Phi;\Theta}\,\mathrm{Cut}$$

$$\frac{\Gamma;\Delta\rhd\bar{a}:A,\Phi;\Theta \mid M:\Diamond A}{\Gamma;\Delta \rhd M \bullet \langle\!|\bar{a}|\!\rangle \lhd \bar{a}:A,\Phi;\Theta}\,\Diamond\Diamond\mathrm{RE} \quad\equiv\quad \frac{\Gamma;\Delta\rhd\bar{a}:A,\Phi;\Theta \mid M:\Diamond A \qquad \frac{\frac{}{\bar{a}:A\mid \Gamma;\Delta\lhd\bar{a}:A,\Phi;\Theta}\,\Diamond\mathrm{IdL}}{\langle\!|\bar{a}|\!\rangle:\Diamond A\mid \Gamma;\Delta\lhd\bar{a}:A,\Phi;\Theta}\,\Diamond\mathrm{L}}{\Gamma;\Delta\rhd M\bullet\langle\!|\bar{a}|\!\rangle\lhd\bar{a}:A,\Phi;\Theta}\,\mathrm{Cut}$$

$$\frac{K:A\mid x:A,\Gamma;\Delta\lhd\Phi;\Theta}{x:A,\Gamma;\Delta \rhd x \bullet K \lhd \Phi;\Theta}\,\mathrm{LE} \quad\equiv\quad \frac{\frac{}{x:A,\Gamma;\Delta\rhd\Phi;\Theta \mid x:A}\,\mathrm{IdR} \qquad K:A\mid x:A,\Gamma;\Delta\lhd\Phi;\Theta}{x:A,\Gamma;\Delta\rhd x\bullet K\lhd\Phi;\Theta}\,\mathrm{Cut}$$

$$\frac{K:A\mid \Gamma;\Delta,a:A\lhd\Phi;\Theta}{\Gamma;\Delta,a:A \rhd a \bullet K \lhd \Phi;\Theta}\,\Box\mathrm{LE} \quad\equiv\quad \frac{\frac{}{\Gamma;\Delta,a:A\rhd\Phi;\Theta \mid a:A}\,\Box\mathrm{IdR} \qquad K:A\mid \Gamma;\Delta,a:A\lhd\Phi;\Theta}{\Gamma;\Delta,a:A\rhd a\bullet K\lhd\Phi;\Theta}\,\mathrm{Cut}$$

$$\frac{K:\Box A\mid \Gamma;\Delta,a:A\lhd\Phi;\Theta}{\Gamma;\Delta,a:A \rhd [\![a]\!] \bullet K \lhd \Phi;\Theta}\,\Box\Box\mathrm{LE} \quad\equiv\quad \frac{\frac{\frac{}{\Gamma;\Delta,a:A\rhd\Phi;\Theta \mid a:A}\,\Box\mathrm{IdR}}{\Gamma;\Delta,a:A\rhd\Phi;\Theta \mid [\![a]\!]:\Box A}\,\Box\mathrm{R} \qquad K:\Box A\mid \Gamma;\Delta,a:A\lhd\Phi;\Theta}{\Gamma;\Delta,a:A\rhd M\bullet\langle\!|\bar{a}|\!\rangle\lhd\Phi;\Theta}\,\mathrm{Cut}$$

Figure 5: Derived structural rules

Prime types	$A', B' ::= X \mid \Diamond A$
Patterns	$v, w ::= x \mid \llbracket a \rrbracket \mid [\bar{x}]\mathrm{not} \mid \langle v, w\rangle \mid \langle v\rangle\mathrm{inl} \mid \langle w\rangle\mathrm{inr}$
Call-by-value coterm	$K, L ::= \bar{x} \mid \bar{a} \mid \langle\!\langle K\rangle\!\rangle \mid M_1.(S_1) \talloblong \cdots \talloblong M_m.(S_m)$
Call-by-value classical antecedent	$\Gamma' ::= x_1 : A'_1, \ldots, x_m : A'_m$
Call-by-value right sequent	$\Gamma' \,;\, \Delta \rhd \Phi \,;\, \Theta \mid M : A$
Call-by-value left sequent	$K : A \mid \Gamma' \,;\, \Delta \lhd \Phi \,;\, \Theta$
Call-by-value center sequent	$\Gamma' \,;\, \Delta \rhd S \lhd \Phi \,;\, \Theta$

$$\frac{}{x : A', \Gamma' \,;\, \Delta \rhd \Phi \,;\, \Theta \mid x : A'}\ \mathrm{IdR} \qquad \frac{}{\bar{x} : A \mid \Gamma' \,;\, \Delta \lhd \Phi \,;\, \Theta, \bar{x} : A}\ \mathrm{IdL}$$

$$\frac{}{\Gamma' \,;\, \Delta, a : A \rhd \Phi \,;\, \Theta \mid a : A}\ \Box\mathrm{IdR} \qquad \frac{}{\bar{a} : A \mid \Gamma' \,;\, \Delta \lhd \bar{a} : A, \Phi \,;\, \Theta}\ \Diamond\mathrm{IdL}$$

$$\frac{\cdot \,;\, \Delta \rhd \Phi \,;\, \cdot \mid M : A}{\Gamma' \,;\, \Delta \rhd \Phi \,;\, \Theta \mid \llbracket M \rrbracket : \Box A}\ \Box\mathrm{R} \qquad \frac{K : A \mid \cdot \,;\, \Delta \lhd \Phi \,;\, \cdot}{\langle\!\langle K\rangle\!\rangle : \Diamond A \mid \Gamma' \,;\, \Delta \lhd \Phi \,;\, \Theta}\ \Diamond\mathrm{L}$$

$$\frac{\Gamma' \,;\, \Delta \rhd \Phi \,;\, \Theta \mid M : A \quad \Gamma' \,;\, \Delta \rhd \Phi \,;\, \Theta \mid N : B}{\Gamma' \,;\, \Delta \rhd \Phi \,;\, \Theta \mid \langle M, N\rangle : A \,\&\, B}\ \&\mathrm{R}$$

Derived rules for &L:

$$\begin{aligned} \mathrm{fst}[K] &\equiv \langle v_1, w_1\rangle.(v_1 \bullet K) \talloblong \cdots \talloblong \langle v_m, w_m\rangle.(v_m \bullet K) \\ \mathrm{snd}[L] &\equiv \langle v_1, w_1\rangle.(w_1 \bullet L) \talloblong \cdots \talloblong \langle v_n, w_n\rangle.(w_n \bullet L) \end{aligned}$$

$$\frac{\Gamma' \,;\, \Delta \rhd \Phi \,;\, \Theta \mid M : A}{\Gamma' \,;\, \Delta \rhd \Phi \,;\, \Theta \mid \langle M\rangle\mathrm{inl} : A \vee B} \qquad \frac{\Gamma' \,;\, \Delta \rhd \Phi \,;\, \Theta \mid N : B}{\Gamma' \,;\, \Delta \rhd \Phi \,;\, \Theta \mid \langle N\rangle\mathrm{inr} : A \vee B}\ \vee\mathrm{R}$$

Derived rule for ∨L:

$$\begin{aligned} [K, L] \equiv\ & \langle v_1\rangle\mathrm{inl}.(v_1 \bullet K) \talloblong \cdots \talloblong \langle v_m\rangle\mathrm{inl}.(v_m \bullet K) \\ & \talloblong \langle w_1\rangle\mathrm{inr}.(w_1 \bullet L) \talloblong \cdots \talloblong \langle w_n\rangle\mathrm{inr}.(w_n \bullet L) \end{aligned}$$

$$\frac{K : A \mid \Gamma' \,;\, \Delta \lhd \Phi \,;\, \Theta}{\Gamma' \,;\, \Delta \rhd \Phi \,;\, \Theta \mid [K]\mathrm{not} : \neg A}\ \neg\mathrm{R}$$

Derived rule for ¬L:

$$\mathrm{not}\langle M\rangle \equiv [\bar{x}]\mathrm{not}.(M \bullet \bar{x})$$

$$\frac{\Gamma' \,;\, \Delta \rhd S \lhd \Phi \,;\, \Theta, \bar{x} : A}{\Gamma' \,;\, \Delta \rhd \Phi \,;\, \Theta \mid (S).\bar{x} : A}\ \mathrm{RI} \qquad \frac{\Gamma'_1, \Gamma'_2 \,;\, \Delta_1, \Delta_2 \rhd S_i \lhd \Phi \,;\, \Theta_1, \Theta_2 \quad \text{for each} \quad \Gamma'_2 \,;\, \Delta_2 \rhd \cdot \,;\, \Theta_2 \mid v_i : A}{v_1.(S_1) \talloblong \cdots \talloblong v_m.(S_m) : A \mid \Gamma'_1 \,;\, \Delta_1 \lhd \Phi \,;\, \Theta_1}\ \mathrm{LI}$$

$$\frac{\Gamma' \,;\, \Delta \rhd S \lhd \bar{a} : A, \Phi \,;\, \Theta}{\Gamma' \,;\, \Delta \rhd \Phi \,;\, \Theta \mid (S).\langle\!\langle \bar{a}\rangle\!\rangle : \Diamond A}\ \Diamond\mathrm{RI}$$

Derived rule for □LI:

$$\llbracket a \rrbracket.(S) \equiv \llbracket a \rrbracket.(S)$$

$$\frac{\Gamma' \,;\, \Delta \rhd \Phi \,;\, \Theta \mid M : A \quad K : A \mid \Gamma' \,;\, \Delta \lhd \Phi \,;\, \Theta}{\Gamma' \,;\, \Delta \rhd M \bullet K \lhd \Phi \,;\, \Theta}\ \mathrm{Cut}$$

Figure 6: Call-by-value calculus

Coprime types	$A^{\backprime}, B^{\backprime} ::= X \mid \square A$
Copatterns	$p, q ::= \bar{x} \mid \langle\!|\bar{a}|\!\rangle \mid \mathrm{not}\langle x \rangle \mid [p, q] \mid \mathrm{fst}[p] \mid \mathrm{snd}[q]$
Call-by-name term	$M, N ::= x \mid a \mid \llbracket M \rrbracket \mid (S_1).K_1 \talloblong \cdots \talloblong (S_m).K_m$
Call-by-name classical succeedent	$\Theta^{\backprime} ::= \bar{y}_1 : B_1^{\backprime}, \ldots, \bar{y}_m : B_m^{\backprime}$
Call-by-name right sequent	$\Gamma ; \Delta \rhd \Phi ; \Theta^{\backprime} \mid M : A$
Call-by-name left sequent	$K : A \mid \Gamma ; \Delta \lhd \Phi ; \Theta^{\backprime}$
Call-by-name center sequent	$\Gamma ; \Delta \rhd S \lhd \Phi ; \Theta^{\backprime}$

$$\frac{}{x : A, \Gamma ; \Delta \rhd \Phi ; \Theta^{\backprime} \mid x : A}\ \mathrm{IdR} \qquad \frac{}{\bar{x} : A^{\backprime} \mid \Gamma ; \Delta \lhd \Phi ; \Theta^{\backprime}, \bar{x} : A^{\backprime}}\ \mathrm{IdL}$$

$$\frac{}{\Gamma ; \Delta, a : A \rhd \Phi ; \Theta^{\backprime} \mid a : A}\ \square\mathrm{IdR} \qquad \frac{}{\bar{a} : A \mid \Gamma ; \Delta \lhd \bar{a} : A, \Phi ; \Theta^{\backprime}}\ \Diamond\mathrm{IdL}$$

$$\frac{\cdot ; \Delta \rhd \Phi ; \cdot \mid M : A}{\Gamma ; \Delta \rhd \Phi ; \Theta^{\backprime} \mid \llbracket M \rrbracket : \square A}\ \square\mathrm{R} \qquad \frac{K : A \mid \cdot ; \Delta \lhd \Phi ; \cdot}{\langle\!| K |\!\rangle : \Diamond A \mid \Gamma ; \Delta \lhd \Phi ; \Theta^{\backprime}}\ \Diamond\mathrm{L}$$

Derived rule for &R:

$$\begin{aligned} \langle M, N \rangle \ \equiv\ & (M \bullet p_1).\mathrm{fst}[p_1] \talloblong \cdots \talloblong (M \bullet p_m).\mathrm{fst}[p_m] \\ & \talloblong (N \bullet q_1).\mathrm{snd}[q_1] \talloblong \cdots \talloblong (N \bullet q_n).\mathrm{snd}[q_n] \end{aligned}$$

$$\frac{K : A \mid \Gamma ; \Delta \lhd \Phi ; \Theta^{\backprime}}{\mathrm{fst}[K] : A \,\&\, B \mid \Gamma ; \Delta \lhd \Phi ; \Theta^{\backprime}} \qquad \frac{L : B \mid \Gamma ; \Delta \lhd \Phi ; \Theta^{\backprime}}{\mathrm{snd}[L] : A \,\&\, B \mid \Gamma ; \Delta \lhd \Phi ; \Theta^{\backprime}}\ \&\mathrm{L}$$

Derived rules for $\vee$R:

$$\begin{aligned} \langle M \rangle\mathrm{inl} \ &\equiv\ (M \bullet p_1).[p_1, q_1] \talloblong \cdots \talloblong (M \bullet p_m).[p_m, q_m] \\ \langle N \rangle\mathrm{inr} \ &\equiv\ (N \bullet q_1).[p_1, q_1] \talloblong \cdots \talloblong (N \bullet q_n).[p_n, q_n] \end{aligned}$$

$$\frac{K : A \mid \Gamma ; \Delta \lhd \Phi ; \Theta^{\backprime} \qquad L : B \mid \Gamma ; \Delta \lhd \Phi ; \Theta^{\backprime}}{[K, L] : A \vee B \mid \Gamma ; \Delta \lhd \Phi ; \Theta^{\backprime}}\ \vee\mathrm{L}$$

Derived rule for $\neg$R:

$$[K]\mathrm{not} \ \equiv\ (x \bullet K).\mathrm{not}\langle x \rangle$$

$$\frac{\Gamma ; \Delta \rhd \Phi ; \Theta^{\backprime} \mid M : A}{\mathrm{not}\langle M \rangle : \neg A \mid \Gamma ; \Delta \lhd \Phi ; \Theta^{\backprime}}\ \neg\mathrm{L}$$

$$\frac{\Gamma_1, \Gamma_2 ; \Delta \rhd S_i \lhd \Phi_1, \Phi_2 ; \Theta_1^{\backprime}, \Theta_2^{\backprime} \quad \text{for each} \quad p_i : A \mid \Gamma_2 ; \cdot \lhd \Phi_2 ; \Theta_2^{\backprime}}{\Gamma_1 ; \Delta \rhd \Phi_1 ; \Theta_1^{\backprime} \mid (S_1).p_1 \talloblong \cdots \talloblong (S_m).p_m : A}\ \mathrm{RI} \qquad \frac{x : A, \Gamma ; \Delta \rhd S \lhd \Phi ; \Theta^{\backprime}}{x.(S) : A \mid \Gamma ; \Delta \lhd \Phi ; \Theta^{\backprime}}\ \mathrm{LI}$$

Derived rule for $\Diamond$RI:

$$(S).\langle\!|\bar{a}|\!\rangle \ \equiv\ (S).\langle\!|\bar{a}|\!\rangle$$

$$\frac{\Gamma ; \Delta, a : A \rhd S \lhd \Phi ; \Theta^{\backprime}}{\llbracket a \rrbracket.(S) : \square A \mid \Gamma ; \Delta \lhd \Phi ; \Theta^{\backprime}}\ \square\mathrm{LI}$$

$$\frac{\Gamma ; \Delta \rhd \Phi ; \Theta^{\backprime} \mid M : A \qquad K : A \mid \Gamma ; \Delta \lhd \Phi ; \Theta^{\backprime}}{\Gamma ; \Delta \rhd M \bullet K \lhd \Phi ; \Theta^{\backprime}}\ \mathrm{Cut}$$

Figure 7: Call-by-name calculus

$$
\begin{array}{ll}
\text{Values} & V, W ::= x \mid \llbracket M \rrbracket \mid \langle V, W\rangle \mid \langle V\rangle\mathrm{inl} \mid \langle W\rangle\mathrm{inr} \mid [K]\mathrm{not} \mid (S).\llparenthesis \bar{a} \rrparenthesis \\
\text{Term context} & E ::= \langle\{\,\}, M\rangle \mid \langle V, \{\,\}\rangle \mid \langle\{\,\}\rangle\mathrm{inl} \mid \langle\{\,\}\rangle\mathrm{inr}
\end{array}
$$

Pattern substitution

$$
\begin{aligned}
S\{\llbracket M \rrbracket / \llbracket a \rrbracket\} &\equiv S\{M/a\} \\
S\{[K]\mathrm{not}/[\bar{x}]\mathrm{not}\} &\equiv S\{K/\bar{x}\} \\
S\{\langle V, W\rangle / \langle v, w\rangle\} &\equiv S\{V/v\}\{W/w\} \\
S\{\langle V\rangle\mathrm{inl}/\langle v\rangle\mathrm{inl}\} &\equiv S\{V/v\} \\
S\{\langle W\rangle\mathrm{inr}/\langle w\rangle\mathrm{inr}\} &\equiv S\{W/w\}
\end{aligned}
$$

$$
\begin{array}{llll}
(\beta L)_v & V \bullet v.(S) \mathbin{\square} \cdots & \Rightarrow_v \; S\{V/v\} & \text{if the substitution } S\{V/v\} \text{ is defined (above)} \\
(\eta L)_v & v_1.(v_1 \bullet K) \mathbin{\square} \cdots \mathbin{\square} v_m.(v_m \bullet K) & \Rightarrow_v \; K & \text{if no variable in any } v_i \text{ appears free in } K \\
(\beta R)_v & (S).\bar{x} \bullet K & \Rightarrow_v \; S\{K/\bar{x}\} & \\
(\eta R)_v & (M \bullet \bar{x}).\bar{x} & \Rightarrow_v \; M & \text{if } \bar{x} \text{ does not appear free in } M \\
(\beta R\Diamond)_v & (S).\llparenthesis \bar{a} \rrparenthesis \bullet \llparenthesis K \rrparenthesis & \Rightarrow_v \; S\{K/\bar{a}\} & \\
(\eta R\Diamond)_v & (M \bullet \llparenthesis \bar{a} \rrparenthesis).\llparenthesis \bar{a} \rrparenthesis & \Rightarrow_v \; M & \text{if } \bar{a} \text{ does not appear free in } M \\
(\varsigma L)_v & E\{M\} & \Rightarrow_v \; (M \bullet v_1.(E\{v_1\} \bullet \bar{y}) \mathbin{\square} \cdots \mathbin{\square} v_m.(E\{v_m\} \bullet \bar{y})).\bar{y} & \text{if } M \text{ is not a value}
\end{array}
$$

Figure 8: Call-by-value reductions

$$
\begin{array}{ll}
\text{Covalues} & P, Q ::= \bar{x} \mid \llparenthesis K \rrparenthesis \mid [V, W] \mid \mathrm{fst}[V] \mid \mathrm{snd}[W] \mid \mathrm{not}\langle M\rangle \mid \llbracket a \rrbracket.(S) \\
\text{Coterm context} & G ::= [\{\,\}, K] \mid [P, \{\,\}] \mid \mathrm{fst}[\{\,\}] \mid \mathrm{snd}[\{\,\}]
\end{array}
$$

Copattern substitution

$$
\begin{aligned}
S\{\llparenthesis K \rrparenthesis / \llparenthesis \bar{a} \rrparenthesis\} &\equiv S\{K/\bar{a}\} \\
S\{\mathrm{not}\langle M\rangle/\mathrm{not}\langle x\rangle\} &\equiv S\{M/x\} \\
S\{[P, Q]/\langle p, q\rangle\} &\equiv S\{P/p\}\{Q/q\} \\
S\{\mathrm{fst}[P]/\mathrm{fst}[p]\} &\equiv S\{P/p\} \\
S\{\mathrm{snd}[Q]/\mathrm{snd}[q]\} &\equiv S\{Q/q\}
\end{aligned}
$$

$$
\begin{array}{llll}
(\beta L)_n & M \bullet x.(S) & \Rightarrow_n \; S\{M/x\} & \\
(\eta L)_n & x.(x \bullet K) & \Rightarrow_n \; K & \text{if } x \text{ does not appear free in } K \\
(\beta R)_n & (S).p \mathbin{\square} \cdots \bullet P & \Rightarrow_n \; S\{P/p\} & \text{if the substitution } S\{P/p\} \text{ is defined (above)} \\
(\eta R)_n & (M \bullet p_1 K).p_1 \mathbin{\square} \cdots \mathbin{\square} (M \bullet p_m).p_m & \Rightarrow_n \; M & \text{if no covariable in any } p_i \text{ appears free in } M \\
(\beta L\square)_n & \llbracket M \rrbracket \bullet \llbracket a \rrbracket.(S) & \Rightarrow_n \; S\{M/a\} & \\
(\eta L\square)_n & \llbracket a \rrbracket.(\llbracket a \rrbracket \bullet K) & \Rightarrow_n \; K & \text{if } a \text{ does not appear free in } K \\
(\varsigma L)_n & G\{K\} & \Rightarrow_n \; y.((y \bullet G\{p_1\}).p_1 \mathbin{\square} \cdots \mathbin{\square} (y \bullet G\{p_m\}).p_m \bullet K) & \text{if } K \text{ is not a covalue}
\end{array}
$$

Figure 9: Call-by-name reductions

unordered bag of copatterns p_i in η-long normal form, paired with statements S_i. See Figure 7 for details.

The modifications to Wadler's dual calculus outlined in the previous two paragraphs are needed to ensure confluence and extensional $\beta\eta$-equality, but not related to the addition of modality. This technique of expanding (co)patterns in binding constructs to η-long normal form seems equivalent to the way Ghani (1995) obtains $\beta\eta$-equality for a λ-calculus with coproducts.

It is straightforward to generalize these calculi to multiple S4-modalities that are situated with respect to each other in a partially ordered set of implications. In addition to the $\Diamond$RI and $\Box$LI typing rules shown in these figures, various *communication rules* may be added to such multiple-S4 calculi to model whether a direct channel is present from one agent to another. It is such communication rules as $\mathsf{A}/\mathsf{B}\ \Box$LI and $\mathsf{A}/\mathsf{C}\ \Box$LI that enable us to model Alice's scheduling plan in Figure 1.

V. RELATED AND FUTURE WORK

Basic meta-theoretic properties like confluence and extensional $\beta\eta$-equality are designed for but remain to be proven.

Jia and Walker (2003, 2004) model distributed computation using modal logic, by treating intuitionistic modal proofs as distributed programs. Their modality models spatial connectivity among network locations. By contrast, this paper uses a classical S4 modality that models epistemic possibility among knowledge states. They give an operational semantics that makes network communication explicit: transitions take place locally at individual places on the network, rather than globally by rewriting an entire proof term as done here. Such an operational semantics would be useful for the logic in this paper as well: a proof of a modal formula like $\mathsf{A}\,\Box X$ may correspond to a name (or a remote pointer) in a process calculus.

The need to model spatial connectivity alongside epistemic possibility (not to mention actions and time) calls for constructive modal logics with Kripke semantics that uniformly accommodate multiple modalities with different axioms. Constructive formulations of labeled deduction and of hybrid logic (Braüner and de Paiva 2003) appear promising in this regard.

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